GRAVITATION

FRANK HARRIS

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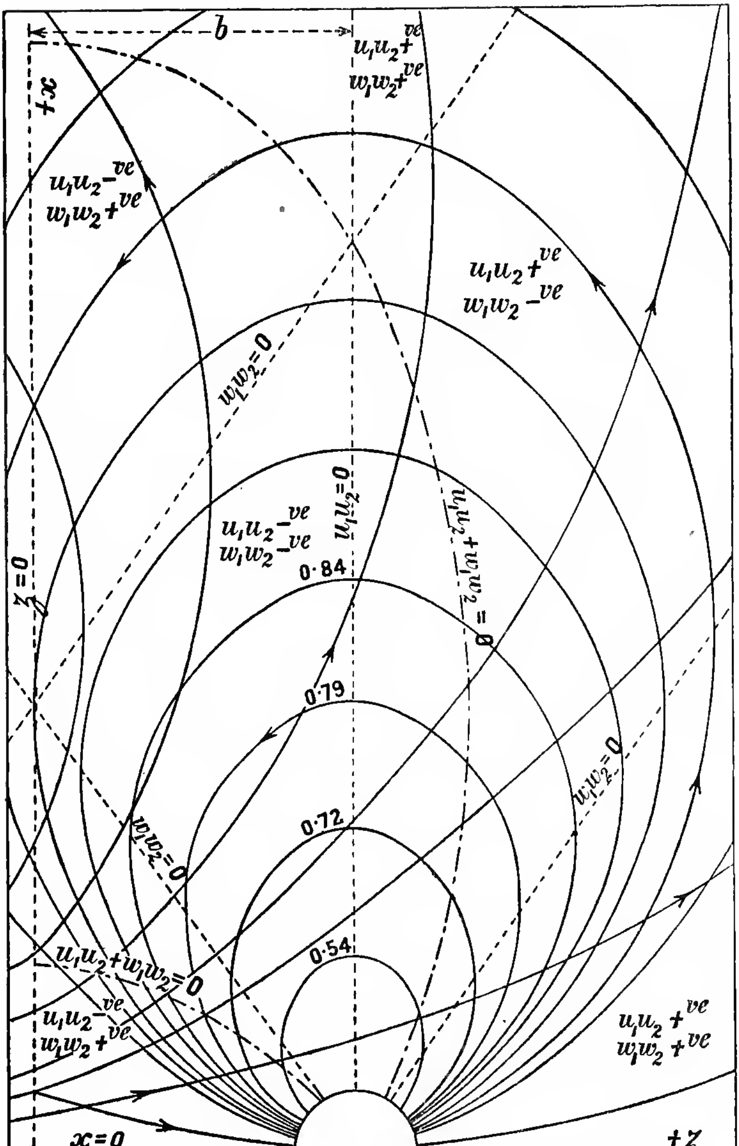

Fig. 1.—Spheres in Sequence: Solitary Lines of Flow.

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E.V.

PREFACE

THE following pages are, in all probability, not entirely free from clerical errors; and possibly involve erroneous deductions.

My only hope is that the germ or two of truth they do contain may win from Fortune access to some better brain than mine.

F. H.

LELINORA,
WHITEBROOK, 1912.

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INTRODUCTION

Before seeking to explain any natural phenomenon it is surely well to understand what we mean by the word explanation.

The complete and full explanation of any single event would probably include that of every other natural phenomenon, and necessarily require knowledge and intelligence to be alike infinite.

It is therefore a subject for congratulation that the complete explanation of each physical property of matter appears to be expressible in the form of a convergent series: so that, although unable to exactly sum the series, we can in most cases evaluate the first few terms; and thus obtain an approximate solution of our problem, which will usually be found sufficient for any immediately concrete object, whether that object be connected with the adaptation of Nature's resources to the wants of man, or that more abstract benefit which is to be found in any extension of our knowledge concerning truth.

So far as experience teaches, it would appear that,—while the essential principles underlying any explanation are perfectly simple,—the details of any actual phenomenon in nature are infinitely complex.

If, being promised all the necessary data, we were asked to determine the motion of a single particle of water on the surface of the Pacific Ocean; what chance would we have of evaluating the resultant effect of the attraction exerted by all celestial and terrestrial matter upon the particle, the acceleration due to the pressure of the wind upon the wave's face of which the particle forms a part, the effect of some shark's

fin as it disturbed the water fifteen feet away, or impulse due to the rotation of a ship's propeller having been reversed at some miles distance half an hour ago?

And yet how perfectly obvious it is that whatever the motions may be which are produced individually by each of these,—together with an infinity of other causes,—the actual resultant motion will simply be that which completely complies with every individual requisition!

If the particle were urged on one account to move three feet to the north, and simultaneously urged by some other impulse to move four feet directly east; its actual shift of five feet $NE\frac{3}{4}E$, would surely want no explanation! It has simply complied with each of the two requirements: and if those two requirements should have been for oppositely equal motions, the particle's remaining stationary would quite as fully answer both.

If we understand the meaning of the terms used, there is no explanation needed of the fact that the cube of three is twenty-seven: the fact is a necessary consequence of our definition; or rather is our definition: there can be no question as to why or how it comes to be so.

If we seek the explanation of the rule for extracting the cube root of a number, wherein does the case differ? Simply in the fact that our mind's visual angle may not be sufficient to include the whole object under inspection. Explanation, therefore, here means some device by which this angle may be rendered more obtuse; or we may be enabled to inspect component parts successively.

Considering more and more complex problems, exactly the same principle is seen to be involved; when once completely analysed, phenomena are found to be but mathematical identities; and then no room is left to question 'why': the thing is as essentially a fact as that the square of two is four. These things,—in so far as they can be called things,—are: and would be just the same if there were no material to be used as counters or no intelligence to picture what they mean.

In like way the same truth is applicable to physical nature: that evolution which we indicate by saying that the com-

bination of two atoms of Hydrogen with one atom of Oxygen forms one molecule of water, will,—when we have once dissected the machine,—be seen to be as necessary a result of all our data as the appearance of twenty-seven upon our 'cubing' three.

In the same way, since we define energy measurement to be the product of a constant into the square of a velocity, the 'conservation of energy' is but one subordinate example of Euclid's forty-seventh proposition.

Explanation is therefore the unravelling of a tangled skein; the thread itself requires no explanation.

If mathematical method were sufficiently extended it would be the means of explaining,—shewing the essential necessity of,—all natural phenomena. That in unskilful hands this powerful tool may haply cut the workman's fingers, is no just reason it should rest in rust. Whatever is untrue must ultimately die, but should the smallest gem of truth be brought to light this will beneficently gleam for ever. Or, if impatient with our slow advance we seek to speculate without analysis, let us concern ourselves with 'how'; for to ask why,—as meaning with what object,—is to ignore the obvious truths; that whatsoever is, can be; and whatsoever can be must,—given sufficient opportunity,—event.

GRAVITATION

I

MEDIUM, PARTICLE AND MOTION

If we grant the objective reality of the universe which we imagine to surround us, and of which we appear to form a part; there are two entities of which we have most direct presumptive evidence,—matter and motion.

This 'matter,' using the word in its wider sense, obviously includes some all-pervading medium and a number of isolated fragments existing in that medium. Of these supposed facts we have more direct and convincing evidence than of anything else we suppose to be, excepting only that one solitary fact of which any conscious being can righteously assert himself to be absolutely certain,—his own existence.

Our 'medium' may be what is understood by the expression 'continuous fluid'; or it may be a stream of discontinuous particles; but to all appearance it is all-pervading; for, so far as our experience has gone, there is no space across which communication does not exist. Our fragments, atoms or ultimate particles,—if there be an ultimate anything,—may be some separate entity, concrete individual and permanent; or concrete without being individual, as is the stationary cloud cap of a wind-struck mountain; or may be some volution of the medium,—some vortex ring or end of vortex line; or may, of course, be something unconceived as yet: but whatsoever details may prove true, medium and atom in some form must be, if what we 'see' has objective existence.

In attempting to elucidate any natural phenomenon, it is surely well to first take into consideration the *necessary* consequences of the existence of those entities which are involved

in the problem before us. To elect a definite hypothesis without definite evidence is to choose what is almost certainly untrue: or in absence of all evidence is to choose what must be untrue. Suppositions should therefore be of the most general character possible. Granting our medium, we may call it a fluid while being prepared to modify such expression if necessary; and may suppose it frictionless, since this would at least give us the first term in our explanation's series; while, if necessary, our results could be modified by further investigation: besides, to say there is friction is to assert the existence of smaller details in the system than the smallest of which we have taken account; if, therefore, there is such a thing as an ultimate medium, it must be frictionless.

And whatever the nature of an ultimate atom may be, we can obviously study with advantage the behaviour of a spherical shell, as being the simplest conception of a finite particle; since whatever truths are deducible from the mutual action of a pervading medium, a spherical shell and motion, must find some place in the explanation of all phenomena involving a medium and particles in motion: but all avoidable detail should be at first excluded. Our initial inquiry may therefore well be concerned with the motion of a massless spherical shell in an incompressible frictionless fluid of unit density.

This, in the case of a solitary sphere, has been adequately treated; the 'velocity potential' is $\phi = -\dfrac{va^3 \cos\theta}{2r^2}$; where a is the radius of the sphere and v its linear velocity along the direction from which θ is measured,—which direction we may take as the axis of z; and employ cylindrical coordinates.

The kinetic energy in the fluid at any point r, θ is therefore $\dfrac{v^2a^6}{2r^6}\left\{\dfrac{\sin^2\theta}{4} + \cos^2\theta\right\}$; giving on integration over all space outside the sphere, $\dfrac{v^2a^3\pi}{3}$; or that representing half the mass of the sphere's volume of fluid. This energy is, of course, really spread over the whole space outside the sphere, but enormously condensed towards its neighbourhood.

The equation to a line of flow is $c=\frac{r}{\sin^2\theta}$; where c is the parameter and the distance from the axis of z when $\theta=\frac{\pi}{2}$, where the velocity is parallel to the direction of the sphere's motion and opposed to it in sign.

These lines of flow are the wards of a key to one of Nature's secret places: if, then, we are burglariously inclined, a good impression of their form must surely well be worth obtaining.

They are obviously ovals with the pointed end at the centre of the moving sphere. Their shape is roughly indicated by the closed curves in fig. 1: where, however, no attempt has been made to space them in proportion to the amount of the momentum, much less of the energy, which they enclose: their positions in the diagram being arranged simply with the object of insuring intersections in each of the compartments into which critical lines divide the figure when lines of flow from a second moving sphere are introduced into the field.

If ϕ is the angle between the radius vector and the normal to the curve, we have,—

$$\phi=\tan^{-1}\frac{2\,.\,c\,.\cos\theta\sin\theta}{r}=\cos^{-1}\sqrt{\frac{r}{4c-3r}}.$$

The breadth between two closely adjacent lines of flow whose parameters differ by Δc, is—

$$\Delta c\frac{dr}{dc}\cos\phi=\Delta c\,.\,\frac{r}{c}\,.\,\sqrt{\frac{r}{4c-3r}};$$

therefore the cross area between two surfaces of flow is $\frac{2\pi r^3}{c\sqrt{4c^2-3cr}}\Delta c$; which, multiplied by the speed at any point in the cross section, that is to say by $\frac{va^3}{2r^3}\sqrt{\frac{4c-3r}{c}}$, gives the discharge $\frac{va^3\pi}{c^2}\Delta c$ passing between a pair of adjacent surfaces of flow; the 'surface of flow' being the surface described by the revolution about z of a line of flow; for, of course, flow is not defined by surfaces; but by linear curves.

The total discharge passing the plane $z=0$ is therefore $\pi va^3\int_a^\infty \frac{dc}{c^2}=\pi va^2$; or the volume swept through by the moving sphere in unit time; as of course it should be.

Again; the energy at a point is $\frac{v^2a^6}{8r^6}\left(\frac{4c-3r}{c}\right)$; and this multiplied by the above cross-sectional area and integrated throughout the length of a line of flow, gives the amount of energy enclosed between two adjacent surfaces of flow, or,—

$$2\int_0^{\frac{s}{2}}\frac{v^2a^6\pi}{4r^3}\cdot\frac{\sqrt{4c-3r}}{c^{\frac{5}{2}}}\Delta c\, ds=\frac{v^2a^6\pi\Delta c}{4c^{\frac{5}{2}}}\int_a^c\frac{4c-3r}{r^3\sqrt{c-r}}dr=\frac{v^2a^4\pi}{2c^{\frac{5}{2}}}(c-a)^{\frac{1}{2}}\Delta c:$$

and this, Δc being made infinitesimal, gives on integration to c between a and c, the whole energy enclosed by the surface of flow whose parameter is c; or $\frac{v^2a^3\pi}{3}\left(\frac{c-a}{c}\right)^{\frac{3}{2}}$. If $c=\infty$, we have the total energy as obtained more directly before, $\frac{v^2a^3\pi}{3}$. Also, by assigning suitable values to c, it appears that the smallest line of flow shewn on our diagram, or that whose whole height is but three times the radius of the moving sphere, encloses more than half the total energy; and that three quarters of the whole energy is enclosed by the line of flow $c=6a$.

II

SPHERES IN SEQUENCE

LET us now consider the motion of two spheres in an infinite fluid. Both spheres are on the axis of z; one at $z=-b$ and one at $z=+b$: each moving with a velocity V along this axis.

We will first ignore the effect produced by the obstruction offered by each sphere to the lines of flow produced by the other. If, then, we suppose the lines of flow from one sphere to pass through the other, although each sphere is impervious to the fluid so far as its own motion is concerned, the velocity of every element of the fluid throughout space will be simply the sum of the motions due to each sphere separately. Subsequently allowance can be made for this neglected effect, so that the mutual action between the spheres can be determined with a fairly close approximation to the truth.

All things being symmetrical about the axis of z, we can use cylindrical coordinates; and calling the velocity perpendicular to z due to the sphere 1 at $z=-b$, u_1: that due to the sphere 2 at $z=+b$, u_2; and calling the components parallel to z, w_1 and w_2, we have,—

$$u_1=\frac{Va^3}{2}\cdot\frac{3xz+3xb}{(x^2+z^2+2bz+b^2)^{\frac{5}{2}}}.$$

$$u_2=\frac{Va^3}{2}\cdot\frac{3xz-3xb}{(x^2+z^2-2bz+b^2)^{\frac{5}{2}}}.$$

$$w_1=\frac{-Va^3}{2}\cdot\frac{x^2-2z^2-4bz-2b^2}{(x^2+z^2+2bz+b^2)^{\frac{5}{2}}}.$$

$$w_2=\frac{-Va^3}{2}\cdot\frac{x^2-2z^2+4bz-2b^2}{(x^2+z^2-2bz+b^2)^{\frac{5}{2}}}.$$

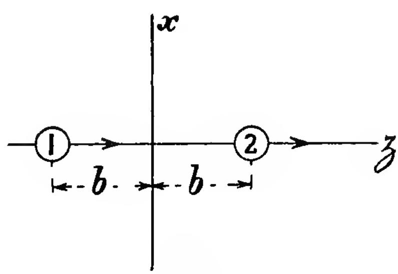

FIG. 2.

The kinetic energy being the integral over all space outside the spheres concerned, of,—

$$\frac{(u_1+u_2)^2+(w_1+w_2)^2}{2}=\frac{u_1{}^2+w_1{}^2}{2}+\frac{u_2{}^2+w_2{}^2}{2}+(u_1u_2+w_1w_2).$$

The first two of these last three items represent the energy which would be due to the two spheres if each were alone. The last, therefore, is the additional energy of each particle of fluid due to the coexistence of the moving spheres.

Say, $$u_1u_2+w_1w_2=\frac{V^2a^6}{4}A\,;$$

where $$A=\frac{x^4+x^2(5z^2-13b^2)+4(z^2-b^2)^2}{\{x^4+2x^2(z^2+b^2)+(z^2-b^2)^2\}^{\frac{5}{2}}}.$$

The annular element having a given velocity being $2\pi x\,dx$ in each plane section perpendicular to z, the integral of A over such planes is,—

$$2\pi\int x\,\frac{x^4+x^2(5z^2-13b^2)+4(z^2-b^2)^2}{\{x^4+2x^2(z^2+b^2)+(z^2-b^2)^2\}^{\frac{5}{2}}}\,dx.$$

Put $m=x^2+b^2+z^2$, $\therefore\ \dfrac{dx}{dm}=\dfrac{1}{2x}$; and we obtain,—

$$\pi\int\frac{m^2+m(3z^2-15b^2)+2b^2z^2+18b^4}{\{m^2-4z^2b^2\}^{\frac{5}{2}}}\,dm.$$

$$=\pi\,\frac{3m^3-18mz^2b^2-2mz^4-4z^6+20b^2z^4}{4z^4\{m^2-4z^2b^2\}^{\frac{3}{2}}},\ \ldots\ldots\ldots\ldots\text{(1)}$$

or, for possible reference translating back into x,—

$$=\pi\,\frac{3x^6+9x^4(b^2+z^2)+x^2(9b^4+7z^4)+3(b^2-z^2)^3}{4z^4\{x^4+2x^2(b^2+z^2)+(z^2-b^2)^2\}^{\frac{3}{2}}}.\ \ldots\text{(2)}$$

The integration with which we are concerned can be more neatly effected by the use of polar coordinates, but this process is more instructive.

For planes between $z=b-a$ and $z=b+a$, or about the sphere at $z=b$, our limits are $x^2=a^2-(z-b)^2$ and $x=\infty$; or $m=a^2+2bz$ and $m=\infty$. Using these either in (1) or (2), we obtain,—

$$\frac{\pi}{a^3}\left\{\frac{z^2}{\{a^2+4bz\}^{\frac{3}{2}}}+\frac{bz}{\{\ ,,\ \}^{\frac{3}{2}}}+\frac{a^2-10b^2}{2\{\ ,,\ \}^{\frac{3}{2}}}+\frac{3b^3}{z\{\ ,,\ \}^{\frac{3}{2}}}\right.$$

$$\left.+\frac{-9a^2b^2}{2z^2\{\ ,,\ \}^{\frac{3}{2}}}+\frac{-9a^4b}{2z^3\{\ ,,\ \}^{\frac{3}{2}}}+\frac{-3a^6}{4z^4\{\ ,,\ \}^{\frac{3}{2}}}+\frac{3a^3}{4z^4}\right\}. \quad \ldots(2a)$$

The integral of this to z is,—

$$\frac{\pi}{a^3}\left\{\frac{a^4}{4z^3(a^2+4bz)^{\frac{1}{2}}}+\frac{ba^2}{2z^2(\ ,,\ )^{\frac{1}{2}}}+\frac{-b^2}{2z(\ ,,\ )^{\frac{1}{2}}}+\frac{30b^4-a^4}{12b^3(\ ,,\ )^{\frac{1}{2}}}\right.$$

$$\left.+\frac{z(3b^2-a^2)}{6b^2(\ ,,\ )^{\frac{1}{2}}}+\frac{z^2}{6b(\ ,,\ )^{\frac{1}{2}}}+\frac{-a^3}{4z^3}\right\}.$$

This taken between the limits $z=b-a$ and $z=b+a$, and multiplied by $\frac{V^2a^6}{4}$, gives the 'additional' energy existing round the sphere between the planes $z=b-a$ and $z=b+a$, or,—

$$\frac{V^2a^6}{4}\int_{b-a}^{b+a}dz\int_{\sqrt{a^2-(z-b)^2}}^{\infty}2\pi xA\,dx=\frac{V^2a^6\pi}{8}\left(-\frac{1}{3b^3}-\frac{1}{(b+a)^3}\right). \quad \ldots(3)$$

By 'additional' energy is meant the excess energy due to the coexistence of the moving spheres, over and above the sum of the energies, over the space in question, which would have existed if it had been possible for the two spheres' energies to exist independently.

For the rest of space to the right of the origin our integral to x, (2), must be taken between the limits 0 to ∞; giving,—

$$\frac{V^2a^6\pi}{4}\left\{\frac{3}{4z^4}-\frac{3(b^2-z^2)^3}{\pm 4z^4(z^2-b^2)^3}=0, \text{ or } \frac{3}{2z^4}\right\}. \quad \ldots\ldots\ldots(4)$$

The double sign in the denominator is in consequence of $(z^2-b^2)^3$ arising from $\{(z^2-b^2)^2\}^{\frac{3}{2}}$.

Now it would seem worth while to pause here in order to get a clear idea of the variation of our function in space. It is true that the question of sign can be more briefly decided; but a clear conception of the physical problem assists in the avoidance of mistakes.

From fig. 1 it is evident that u_1u_2 is negative between $z=0$ and $z=b$; and is positive beyond. The fluid is motionless at $z=0$, $x=\sqrt{2}b$; and the product w_1w_2 is negative

between the cones $x=\sqrt{2}(z\pm b)$, beyond $x=\sqrt{2}(b-z)$. Our whole function $u_1u_2+w_1w_2$ is zero when

$$x^4+x^2(5z^2-13b^2)+4(z^2-b^2)^2=0\,;$$

or when $$x=\pm\left\{\frac{13b^2-5z^2\pm\sqrt{153b^4-98b^2z^2+9z^4}}{2}\right\}^{\frac{1}{2}}\quad\text{.........(5)}$$

The trace of this surface is shewn by the dot and dash line in fig. 1.

The function is obviously positive below the line $x=\sqrt{2}(z-b)$; and as it cannot change sign except by passing across the real portion of (5), it must be wholly negative within this region and wholly positive without.

We see, therefore, that our integral to x must be positive and finite beyond $z=1{\cdot}374b$; and it may be zero between $z=0$ and $z=b-a$.

Also, the value $\frac{3}{2z^4}$ would give an infinite value to the integral where $z=0$; while our function's integral is necessarily everywhere finite. Therefore our integral to x, if of this form, must become zero at $z=0$, lest its integral to z give an infinite value.

Lastly, the denominator of (4) in the bracket simply stands for the product of two radii vectores, essentially positive distances. It is true that this denominator was made up of $r^2\times$denominator of $\cos\theta$; but the denominator of $\cos\theta$ is necessarily positive too. It is the numerator, or z, which changes sign as θ passes $\frac{\pi}{2}$. Any ambiguity in sign, therefore, is an introduced ambiguity due to our method of expressing x and z in terms of r. When retranslating, therefore, we know our root should be positive.

On each count, then, we must take the upper sign. The numerator has not been squared: it naturally changes sign as z passes b; therefore,—our integral between $z=0$ and $z=b-a$ is zero, and beyond $z=b+a$ it is

$$\frac{V^2a^6\pi}{8}\int_{b+a}^{\infty}\frac{3}{z^4}dz=\frac{V^2a^6\pi}{8}\cdot\frac{1}{(b+a)^3}\quad\text{...............(5}a\text{)}$$

Adding this to (3), we have the total additional energy to the right of the origin; or the portion appropriate to each sphere is,—subject to the neglected item,—

$$-\frac{V^2a^6\pi}{3d^3}, \qquad \text{(6)}$$

if we write $2b=d$.

Now the neglected item was simply the obstruction actually offered by one sphere to the current in its neighbourhood due to the other. This can be conveniently divided into four terms; the first, and by far the largest term, is that dependent upon the obstruction offered by the impermeable sphere to a uniform current equal to the mean current parallel to z which exists at the site of the obstructing sphere; the second is the energy found inside the permeable sphere but absent from the impermeable; the third is that small correction which is due to the fact that the effect of a non-uniform current is not identical with the effect of a uniform current of like discharge; and the fourth is that due to the small radial discharge which passed through the sides of the permeable sphere, but is stopped by the impermeable surface.

The first term is obviously the effect we have just been evaluating: that due to a relative motion between sphere and fluid parallel to z; or we have ignored here those components of velocity over all space which would be due to the motion of a sphere in stationary fluid with a velocity equal to the mean velocity parallel to z actually existing round the permeable sphere, with its sign reversed.

The velocity at $z=+b$, due to a sphere at $z=-b$ moving with a velocity V, is $\frac{Va^3}{d^3}$; or say Ve^3. The velocity at every point in space due to this relative motion between sphere and fluid, is that due to a sphere (4) at $z=+b$ moving in stationary fluid with a velocity $-Ve^3$. This, on the same relative assumption as before, is simply $-e^3$ of the velocity due at the same points to the motion of our original sphere (2). That is to say, $u_4=-e^3u_2$ and $w_4=-e^3w_2$. The velocity in the fluid at $z=-b$ due to the sphere (2) is Ve^3. Its effect on the sphere (1) considered as an obstruction, is to produce

velocities in the fluid such as those which would be due to a sphere (3) there, moving with a velocity $-Ve^3$, or $u_3=-e^3u_1$ etc.

The velocities thus obtained as due to these two first correctional spheres, require a similar correction similarly obtained; and so on: or $u_6=e^6u_2$; $u_5=e^6u_1$; $u_8=-e^9u_2$; $u_7=-e^9u_1$ etc.

Instead, therefore, of stating that the kinetic energy at a point in space was $\dfrac{(u_1+u_2)^2+(w_1+w_2)^2}{2}$, we ought to have taken $\dfrac{(u_1+u_2+u_3\ldots u_8)^2+(w_1+w_2\ldots w_8)^2}{2}$.

Multiplying this out, and writing mV for the effective velocity at site of one sphere due to the motion of the other; for Ve^3 being the velocity at the centre was but an approximation to that at the surface; we obtain for the total energy throughout space; both velocity components being represented by u,—

$$\int\frac{u_1^2+u_2^2}{2}=\frac{V^2a^3\pi}{3}\times 2;\qquad \int u_1u_2=-2\frac{V^2a^6\pi}{3d^3}.$$

$$\int\frac{u_3^2+u_4^2}{2}=\quad ,,\quad\times 2m^2;\qquad \int u_1u_4+u_2u_3=\quad ,,\quad\times -2m.$$

$$\int u_1u_3+u_2u_4=\quad ,,\quad\times -4m;\qquad \int u_1u_6+u_2u_5=\quad ,,\quad\times 2m^2.$$

$$\int u_1u_5+u_2u_6=\quad ,,\quad\times 4m^2;\qquad \int u_3u_4=\quad ,,\quad\times m^2.$$

$$\int u_1u_7+u_2u_8+u_3u_5+u_4u_6=\frac{V^2a^3\pi}{3}\times -8m^3.$$

Terms which would result in higher powers of e than the ninth having been omitted; we obtain,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}(1-2m+3m^2-4m^3)-\frac{V^2a^3\pi}{3}e^3(1-2m+3m^2)$$

$$=\frac{V^2a^3\pi}{3}\cdot\frac{1-e^3}{(m+1)^2}\quad\text{.......................(7)}$$

for the energy over half space, two spheres in sequence direct.

If we take the 'central' velocity e^3V for mV; and of course this can be done except when the spheres are within molecular distances of each other; we obtain,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}(1-3e^3+5e^6-7e^9\ldots). \quad\ldots\ldots\ldots\ldots\ldots(8)$$

Giving $$\frac{dE}{dd}=\frac{V^2a^6\pi}{d^4}(6-20e^3+42e^6-72e^9\ldots). \quad\ldots\ldots\ldots\ldots(9)$$

We can sum these series, if we like, and write,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left\{\frac{1-e^3}{(1+e^3)^2}\right\}; \quad \text{and} \quad \frac{dE}{dd}=\frac{V^2a^6\pi}{d^4}2\left\{\frac{3-e^3}{(1+e^3)^2}\right\};$$

but the series are more convenient for most purposes.

In the particular case here considered, since everything is symmetrical about z, all resultant force and motion of spheres must lie along z; there is no ambiguity, and we may determine the meaning of $\frac{dE}{dd}$ directly.

Let F be the force acting between the spheres: not, of course, intrinsic in the spheres themselves, but simply the result of their deflecting the motion of the fluid; and consider F positive if a repulsion. That is to say, $+F$ acts on sphere 2 and $-F$ on sphere 1. Now these forces would accelerate the spheres, while our datum is the differential coefficient of the energy when b alone varies. As external agents, therefore, let us impose equal and opposite forces: that is to say, $-F$ on sphere 2 and $+F$ on sphere 1. The spheres will continue to move at the old velocity V and remain at the old distance apart $2b$. Our force $-F$ being dragged along against us, work is being done upon us by the system at the rate FV: our force $+F$ acting on a point moving in its own direction, does work on the system FV: on the whole no work is done; V, b and E remain constant.

Now for an infinitesimal period of time let us relax our restraining forces, promptly rebalancing as soon as a minute additional velocity v has been added to that of sphere 2 and subtracted from that of sphere 1. This velocity v will serve our purpose however small it may be; hence by making the time of disturbance sufficiently short we may make any effect on the energy during that time, less than any assignable quantity.

Now the distance between the spheres will increase at a constant rate $2v$. The work we are doing on the system will

be at the rate $F(V-v)-F(V+v)=-2Fv$. This, then, must equal the rate at which kinetic energy is increasing in the fluid; since there is no other source of energy present;

or,— $$-2Fv=\frac{dE}{dt}=\frac{dE}{dV}\cdot\frac{dV}{dt}+\frac{dE}{db}\cdot\frac{db}{dt};$$

but we have been careful to arrange that our velocity shall be constant; and since $\frac{db}{dt}=v$, the above equation becomes,

$$F=-\frac{1}{2}\frac{dE}{db}=-\frac{dE}{dd}. \qquad (10)$$

In the particular case we have been considering, $\frac{dE}{dd}$ is a positive quantity; so the force is an attraction; which, we see from (9), varies inversely as the fourth power of the distance when that distance is great compared with the radius of the spheres, being modified, however, when the particles are near together.

It may be worth noting that, during our integration to determine the 'additional energy,' we ascertained that this function was zero when taken over any plane perpendicular to z within the infinite disc comprised between $z=-b+a$ and $z=+b-a$. If, instead of using rectangular cylindrical coordinates, we had used polar coordinates; we would have found that the same function was zero when taken over any spherical surface with a moving sphere as centre, provided that surface lay within the spherical shell comprised between $r=a$ and $r=d-a$. Both these facts have, of course, their physical interpretation.

The largest term in the correction still to be made in our value of E, is that due to the fact that the mean velocity to which our 'correctional spheres' are exposed is not that which is appropriate to the position of their centres.

To obtain this mean velocity we want the integral discharge passing through the permeable sphere: that is to say, of w; the component parallel to z; over the surface $x^2=a^2-(d-z)^2$: the origin being at the centre of the moving sphere.

We have $w=-\frac{Va^3}{2}\frac{x^2-2z^2}{(x^2+z^2)^{\frac{5}{2}}}$; and $\frac{dx}{dz}=\frac{d-z}{x}$.

In integrating for this discharge the area must be taken as positive. The velocity positive or negative is, of course, to be summed; but the area in computing a discharge is essentially positive.

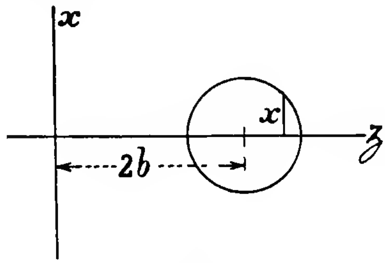

FIG. 3.

Quite independently, therefore, of any change in the variable from x to z, we must integrate in that direction in which x increases, for our element of area is $2\pi x dx$; and as x is essentially positive in cylindrical coordinates, dx must be made so too.

Discharge is, therefore, for the near side and for the far side, in each case,—

$$\frac{-Va^3}{2}\int_{x=0}^{x=a}\frac{x^2-2z^2}{(x^2+z^2)^{\frac{5}{2}}}2\pi x dx;$$

or, changing our variable to z,—

$$-Va^3\pi\int_{d-a\,;\,d+a}^{d}\frac{3z^3+z^2(-5d)+z(3d^2-a^2)+da^2-d^3}{(2dz+a^2-d^2)^{\frac{5}{2}}}dz$$

$$=-Va^3\pi\{2\phi(d)-\phi(d-a)-\phi(d+a)\},$$

where $$\phi(z)=\frac{z^3}{d\{2dz+a^2-d^2\}^{\frac{3}{2}}}+\frac{z^2(-3a^2-2d^2)}{d^2\{\ ,,\ \}^{\frac{3}{2}}}$$

$$+\frac{z(-6a^4+3a^2d^2+d^4)}{d^3\{\ ,,\ \}^{\frac{3}{2}}}+\frac{-2a^6+3a^4d^2-a^2d^4}{d^4\{\ ,,\ \}^{\frac{3}{2}}}.$$

Now

$$\phi(d+a)=\frac{-2a^3}{d^4};\ \phi(d-a)=\frac{+2a^3}{d^4};\ \text{and}\ \phi(d)=\frac{a^2(-2a^2-d^2)}{d^4(a^2+d^2)^{\frac{1}{2}}}.$$

$$\therefore\ \text{discharge}=2Va^3\pi\frac{a^2(d^2+2a^2)}{d^4(d^2+a^2)^{\frac{1}{2}}}$$

and mean velocity

$$= Ve^3\frac{1+2e^2}{(1+e^2)^{\frac{1}{2}}} = Ve^3\left\{1+\frac{3}{2}e^2-\frac{5}{8}e^4+\frac{7}{16}e^6\ldots\right\}\text{.........}(11)$$

Now in passing it may be remarked that although d cannot be less than $2a$ if our spherical shells cannot penetrate each other, it will serve as a check upon our method if we consider the extreme case of the supposed penetration of one sphere by the other until they coincide.

Since the lines of flow due to its own motion do not penetrate a moving sphere, that portion, of the sphere over which

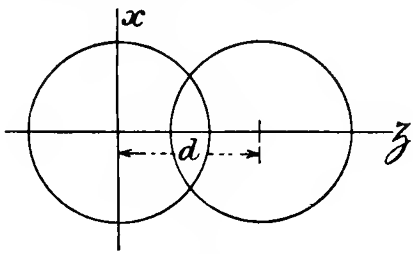

FIG. 4.

we are evaluating the discharge, which lies inside the moving sphere, is exposed to no fluid pressure at all; the 'discharge' there is nil: our summation therefore begins at $z=\frac{d}{2}$; and the total discharge now is,—

$$-Va^3\pi\left\{2\phi(d)-\phi\left(\frac{d}{2}\right)-\phi(d+a)\right\};$$

$\phi(z)$ being as before, and $\phi\left(\frac{d}{2}\right)=\frac{d^6-2d^4a^2-16a^6}{8d^4a^3}$,

or $\quad$ discharge $= -Va^3\pi\left\{\frac{-d^6+2d^4a^2+32a^6}{8d^4a^2}+\frac{-4a^4-2a^2d^2}{d^4(a^2+d^2)^{\frac{1}{2}}}\right\};$

and, as we wrote $\frac{a}{d}=e$, now writing $\frac{d}{a}=g$, we have

$$\text{mean velocity} = V\left\{\frac{g^2-2-32e^4}{16}+\frac{2e^4+e^2}{(1+g^2)^{\frac{1}{2}}}\right\}$$

$$= V\left\{\frac{1}{8}-\frac{3}{16}g^2+\frac{15}{64}g^4-\frac{7}{32}g^6, \text{ etc.}\right\}$$

for spheres in sequence during the latter half of penetration.

Now when the two spherical surfaces are just going to coincide, the discharge is passing through one hemisphere only, of the sphere of integration. Immediately the spheres do coincide the 'discharge' either comes up to, or starts from the whole of the surface,—but passes through none of it. As we are concerned with the impact or pressure of the flow as against an impenetrable surface, the 'mean velocity' as above calculated must be doubled *per saltum* at the moment of coincidence; or, when $g=0$,—

$$\text{mean velocity} = \frac{V}{4}.$$

This does not mean that our function is discontinuous, but simply that we have been evaluating the discharge as passing through the surface; so that when it simply starts from the surface half values are obtained. Our final result could, of course, have been obtained much more simply by putting $d=0$ in the original function before integration.

Now before substituting this in (7) we must remember that in evaluating our 'additional energy over half space' we integrated over sphere 2 between limits $z=b-a$ and $z=b+a$; but if $b=0$ this would lie half in the negative quadrant; so our required limits now are $z=0$ and $z=a$.

In the value (2a) of our integral to x over the sphere, if we put $b=0$ we obtain

$$\frac{\pi}{a^6}\left\{z^2+\frac{a^2}{2}\right\};$$

integral of which to z is

$$\frac{\pi}{a^6}\left(\frac{z^3}{3}+\frac{a^2 z}{2}\right);$$

giving between $z=0$ and $z=a$; $\frac{\pi}{a^3}\left(-\frac{5}{6}\right)$: and since the value beyond the sphere is as before, we have for the additional energy over half space $\frac{V^2a^3\pi}{3}\left(-\frac{1}{4}\right)$. Substituting these values in (7) we obtain for the case of coincidence,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left(\frac{1}{2}-\frac{1}{50}\right). \qquad \text{(12)}$$

Now it may have been noticed in our original assertion that the kinetic energy due to the coexistence of two moving spheres was the integral of $\frac{(u_1+u_2)^2+(w_1+w_2)^2}{2}$, that, in consequence, if the two spheres could move up until they coincided, the energy would be given as four times that due to a single sphere alone. But this is quite what it should be under our first assumption; for to suppose the lines of flow of each sphere pass unobstructed through the other, and yet each in its own motion detrudes the fluid in front of it, is to suppose that if they coincided their detruding effect would be coincident and coexistent; and hence the resultant velocity doubled at all points. It follows, however, that when we profess to have, with fair approximation, corrected the error due to this assumption, the resulting energy should, in such an extreme case, be that due to a single sphere alone.

Referring, then, to (12) it appears that the small fraction $\frac{1}{50}$ represents the resultant sum of all neglected errors in this extreme case: and both our neglected errors here have their violently accentuated maxima. The velocity perpendicular to z, which is exceedingly minute when d is large, becomes great when $d=0$. And our assumption that the effect of an uneven current upon an obstacle is the effect of a uniform current having the mean velocity of the uneven current, fairly true when d is large, is far from right when d is zero; where, within the limits of each hemisphere the elements of discharge even change their sign.

Now, applying our mean velocity (11) in place of Vm in (7) we obtain,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left(1-3e^3-3e^5+5e^6+\frac{5}{4}e^7+12e^8-7\tfrac{7}{8}e^9\right). \quad \ldots(13)$$

The next correction is for the energy existing inside a permeable sphere, which is excluded when we render the sphere impermeable.

The energy due to a moving sphere at the origin, taken over a plane perpendicular to z enclosed within the surface $x^2=a^2-(z-d)^2$; is,—

$$\frac{V^2a^6\pi}{4}\int_0^{\sqrt{a^2-(z-d)^2}}\left(\frac{x}{(x^2+z^2)^3}+\frac{3z^2x}{(x^2+z^2)^4}\right)dx$$

$$=\frac{V^2a^6\pi}{16}\left\{\frac{3}{z^4}+\frac{-1}{(a^2-d^2+2dz)^2}+\frac{-2z^2}{(\;,,\;)^3}\right\};$$

integral of which to z is,

$$\frac{V^2a^6\pi}{16}\left\{\frac{-1}{z^3}+\frac{z^2}{2d(a^2-d^2+2dz)^2}+\frac{z+d}{2d^2(\;,,\;)}-\frac{\log(a^2-d^2+2dz)}{4d^3}\right\}.$$

Value of this between $z=d-a$ and $z=d+a$; is,—

$$\frac{V^2a^6\pi}{16}\left\{\frac{ad^4+8d^2a^3-a^5}{d^2(d^2-a^2)^3}-\frac{1}{2d^3}\log\frac{d+a}{d-a}\right\}$$

$$=\frac{V^2a^3\pi}{3}\left\{2e^6+\frac{27}{5}e^8,\text{ etc.}\right\};\;\ldots\ldots(13a)$$

ignoring higher powers of e than the ninth.

This amount ought to have been subtracted from the value $\frac{V^2a^3\pi}{3}$ to which we equated $\int u_1^2$, etc., in obtaining equation (7). This means a correction applicable to (13) amounting to,—

$$\frac{V^2a^3\pi}{3}\left\{-2e^6-\frac{27}{5}e^8\right\}\{1-2e^3\}=\frac{V^2a^3\pi}{3}\left\{-2e^6-\frac{27}{5}e^8+4e^9\right\},\;(14)$$

in which we write $2e^3$ for $2m$, because higher terms would exceed e^9. Applying this we obtain,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left(1-3e^3-3e^5+3e^6+\frac{5}{4}e^7+\frac{33}{5}e^8-3\tfrac{7}{8}e^9\right)\;\ldots(15)$$

and $$\frac{dE}{dd}=+\frac{V^2a^6\pi}{d^4}\left(6+10e^2-12e^3-\frac{35}{6}e^4-35e^5+23e^6\right)\;\ldots(16)$$

for energy and force; spheres in sequence direct: the last three terms being probably in error.

Concerning the remaining corrections. If we refer to the process by which we obtained the mean velocity (11), it is obvious that the real discharge impinging on the near side of sphere 2 is $Va^3\pi\times\frac{2a^3}{d^4}=Va^2\pi 2e^4$ greater than the discharge represented by the mean velocity, and the discharge leaving the far side is less, by the same amount, than that we have allowed for in employing the mean velocity.

Now these two discharges which are stopped by the impermeable sphere, have against the obstruction the same effect over space as would have a reversed discharge from the sphere. That is to say we have still to compound the effect of $2e^4Va^2\pi$ parallel to z leaving the inner side or rear, and a like discharge leaving the outer side or front. That is to say we have a sort of 'z-part source' of discharge $4e^4Va^2\pi$. Call this (α).

Again; we have completely ignored the radial discharge, or that perpendicular to z, which passed through the permeable sphere, but was stopped when we made it impermeable. Now the first term in this, corresponding to that of our main correction for longitudinal velocity at the centre of one sphere due to the motion of the other, is non-existent,—there being no radial velocity at the centre; and the mean velocity in any direction perpendicular to z being zero. There remains the small radial velocity equal and opposite on opposite sides of each meridional circle.

The discharge perpendicular to and away from z, over $x^2=a^2-(z-d)^2$, is,—

$$\frac{Va^3}{2}\int_{z=d-a}^{z=d+a}\frac{3xz}{(x^2+z^2)^{\frac{5}{2}}}2\pi x\,dz=3Va^3\pi\int_{d-a}^{d+a}\frac{-z^3+z^2 2d+z(a^2-d^2)}{(a^2-d^2+2dz)^{\frac{5}{2}}}dz$$

$$=Va^3\pi\Bigg[\frac{-z^3}{d\{a^2-d^2+2dz\}^{\frac{3}{2}}}+\frac{z^2(3a^2+3d^2)}{d^2\{\;,,\;\}^{\frac{3}{2}}}$$

$$+\frac{z(6a^4-3a^2d^2-3d^4)}{d^3\{\;,,\;\}^{\frac{3}{2}}}+\frac{2a^6-3a^4d^2+d^6}{d^4\{\;,,\;\}^{\frac{3}{2}}}\Bigg]_{d-a}^{d+a}$$

$=4e^4Va^2\pi$. Call this (β).

The area concerned being π^2a^2, the mean velocity is $\frac{4Ve^4}{\pi}$: that is to say mean radial velocity: not directional.

The stoppage of this discharge which before issued from the permeable sphere, means an effect over space equivalent to that of a like discharge directed towards the sphere. Or (β) means an 'x-part sink' of discharge $4e^4Va^2\pi$.

Now the areas over which (α) and (β) were the integrals are identical, being the whole surface of the sphere: if they

had been evenly distributed over this area the two discharges would have cancelled. Actually the maximum of (α) is at the poles and the maximum of (β) near the equator.

There remains, therefore, a part source springing from about the poles and an equal part sink arriving about the equator. This means lines of flow four lobed on planes through z.

Now the sink ends of these lines of flow will not compound to any appreciable extent, being almost perpendicular to the main lines of flow. The source ends will fully compound but with opposite signed effect about the opposite poles. We will not, however, concern ourselves with the exact magnitude of this effect, but be content to ascertain the lowest power of e which can be affected.

The solitary energy of any form of flow has necessarily the square of the velocity in all its terms: here the velocity being Ve^4, the solitary energy of this item cannot affect powers of e below the eighth.

Now the order of magnitude in e of the additional energy depends upon the freedom for expansion granted to the lines of flow. Co-natal lines of flow are mutual positive constraints upon each other; each preventing the other from appropriating an unfair proportion of available space. Thus in the case of a *complete* spherical source, each 'tube' of flow has but its own proper cone of space in which to spread: hence, as we shall shortly see, the additional energy in the case of two three-dimensional sources is of the first power of e. If we remove half of the source we double the space available for the remaining lines of flow: but if we substitute a half sink for the half source removed, we add, as it were, a new dimension to space: the perfect expansion before obtainable only by motion to infinity is now obtainable by a journey to the opposite pole. Thus; moving spheres, which have in effect a source at one side and a sink at the other, compounding possess, as we have seen, additional energy in the third power of e. Similarly, a complete sink and a moving sphere compound in the second power of e.

Now the flow we are considering has all the freedom for expansion possessed by that of a moving sphere: if not more.

We have brought the sink nearer to the source and spread it over the tropics instead of concentrating it about the opposite pole. This system, therefore, cannot compound either with moving sphere systems or with its own like, to produce additional energy of a lower power than $VV'e^3$. And as V' here is Ve^4, it is impossible for our neglected items to affect (15) in the sixth or any lower power of e.

We may accept, therefore, as more than sufficiently exact for our present purpose, and probably good enough for molecular as distinguished from atomic distances, both (15) and (16), after discarding their last three terms.

What then is the meaning of the change in sign between the second and fourth terms of the expression for E? The explanation is not far to seek.

If we introduce a boundary, we impose a positive constraint: the fluid has less space in which to circulate, therefore a given circulation requires on the whole higher velocities; hence more energy is required to set the spheres in motion when a boundary exists than when it is absent. Correspondingly it follows that if we add a moving sphere whose lines of flow on the whole are opposed to those of an existing system, we introduce a negative 'constraint'; a given motion of fluid serves partly the requirements of both systems, so the total motion is less than twice that required for one sphere when alone. Hence our additional energy in the case here considered is negative.

But the introduction of the second sphere itself,—apart from the circulation its motion causes,—is the introduction of a boundary shutting off a portion of space; utterly inappreciable in effect when far from the other sphere; but not negligible when it has moved up so close as to bring the deducted portion of space within that region where velocity is high.

We notice that, even when the spheres touch, the positive term cannot reverse the sign of the additional energy: but this is a detail depending upon the circumstance of our present problem; and we see that under other circumstances it is quite possible for an attraction at a distance to become a repulsion at close quarters.

If we reverse the velocity of one sphere, say of sphere 1, we obtain the corresponding results for two spheres moving in opposite directions. The additional energy simply changes sign; u_4, u_3, u_8 and u_7 change sign, and the remaining components are unaltered. We have in consequence,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}(1+2m+3m^2+4m^3)+\frac{V^2a^3\pi}{3}e^3(1+2m+3m^2) \quad (17)$$

for spheres in sequence reversed.

Using the mean velocity and applying the correction for enclosed energy, which, in this case is,—

$$\frac{V^2a^3\pi}{3}\left(-2e^6-\frac{27}{5}e^8\right)(1+2e^3), \dots\dots\dots\dots\dots\dots(18)$$

we have,

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left(1+3e^3+3e^5+3e^6-\frac{5}{4}e^7+\frac{33}{5}e^8+\frac{31}{8}e^9\right), \dots\dots(19)$$

$$\frac{dE}{dd}=\frac{V^2a^6\pi}{d^4}\left(-6-10e^2-12e^3+\frac{35}{6}e^4-\frac{176}{5}e^5-\frac{93}{4}e^6\right).\dots(20)$$

In which, again, we shall do well to ignore the last three terms.

Now the term $-\frac{5}{4}e^7$ has no significance in its sign: it arises simply from our arbitrary 'mean velocity': very possibly it would become positive if we allowed properly for our neglected part sink-sources; anyhow, we see that the significant change of sign between the second and fourth terms in the expression for E when the sequence was direct, has now vanished; as of course it should have done. The whole of the constraint is here positive; being in fact equivalent to a plane boundary $z=0$. Hence it has long been known that two spheres moving in sequence reversed, repel each other. I do not know whether the amount of the force has before been given. It is,—

$$\frac{V^2a^6\pi}{d^4}(6+10e^2+12e^3).$$

III

POTENTIAL ENERGY

When we are concerned with problems in which the direction of motion of a particle is not coincident with that of the force upon it which we wish to determine, or cases in which there is a source of energy involved, or other cause for ambiguity exists, we are not able to determine the sign of the force in that simple and abstract manner which was adequate for the treatment of two spheres in sequence.

In connection with this subject it is convenient to speak of 'potential energy'; and therefore a digression here may possibly be forgiven while we endeavour to make clear what is to be understood by this cruelly ill-used expression.

There is but one energy,—matter in motion: but if the moving matter be confined or constrained by boundaries, that fact in itself implies pressure upon the boundaries: for if these do not deflect the moving mass they do not constrain; and if they do deflect the motion they exert a force to deflect it.

If the boundaries are incapable of yielding, no energy can thus enter or leave the system; but if they do or can yield against force, insomuch does or can energy pass into or out of the system.

If the pressure outside the movable boundary is greater than that within, the system can be said to possess,—in addition to the actual energy existing in it,—the potentiality of more energy; that energy which could enter the system while the movable boundary advanced under the exterior force from its present position up to the extreme limit in which it exercises its maximum constraint upon the motion

in the system. This energy,—which by motion of the movable boundary can enter the system,—is called its potential energy. It therefore follows that the sum of the kinetic and potential energies of any system is a constant quantity.

If the pressure exerted by the motional matter in the system upon its movable boundary is greater than the pressure from outside, the potential energy is negative; for the potentiality now consists in the possibility of energy leaving the system. The constancy of the sum of kinetic and potential energies, of course, still remains unaffected.

The amount of the energy entering or leaving the system, is obviously the integral of the product of the pressure against which the boundary moves into the element of distance moved against the pressure. The boundary is supposed to be massless; if its motion involve the motion of mass, then the energy represented by the motion of that mass belongs to the system or is exterior to it according to the position of the boundary with regard to the mass. If the pressure against which the boundary moves is nil, no energy passes the boundary: but the potential energy means the maximum amount of energy which *can* pass the boundary in virtue of its possibility of motion. In order, therefore, to measure the potential energy we must suppose the acting pressure balanced by an equal resistance; or at least by a resistance infinitesimally less than the active pressure: and which at the limit of summation is supposed equal to it. Time is no element in the problem; or it is supposed available in infinite quantity.

To take a simple concrete example: conceive a cylindrical envelope of unit cross section, closed at one end by a fixed boundary and at unit distance from this by a movable piston. Let the envelope be incapable of vibration or of otherwise absorbing or permitting the passage through it, of motion; but perfectly elastic to reflect any impinging particles: so that it is impossible for energy to pass the boundary except when some portion of it is moving bodily and perpendicularly to itself. If we properly consider vibration and the conduction of heat as local though short-lived motion

of the boundary, we need not except vibration and imperfect elasticity; but to do this would be to add a useless complication in detail, without in any way affecting the principle concerned.

Let this envelope contain n particles, each of unit mass, moving with a constant speed; velocity $\pm V$ parallel to the axis of the cylinder; and suppose these particles either incapable of internal vibration and of rotation, *or* suppose their vibration and molecular rotation constant throughout our experiment. Here, again, these provisos are required only to avoid the useless complication in detail which would result from adding some myriad minute movable 'pistons' to that large fraction of the boundary whose motional effect we propose to consider.

What we call pressure is simply the force which produces change of motion: and is measured by the rate at which change of motion in mass is produced. The number of the particles or units of mass which strike the inside of the piston per second, when this is at distance v from the end, is $\frac{Vn}{2v}$; supposing no time is lost during reflection. The momentum of each is V, and the motion is not only stopped but also reversed; therefore the pressure on the piston is $F=\frac{V^2n}{v}=\frac{2E_v}{v}$; where E_v is the energy inside the envelope at volume v.

Now if the piston were moved out to, say double the volume, at any velocity greater than V; no change would be made in E. The particles which were just about to strike the piston would simply move on another unit distance and, after they had reached the piston's new position, average pressure would be half what it had been before. If the massless piston move with any less velocity than V, it must be restrained by some force applied outside: the retreating boundary checks the velocity of the impinging particles but, yielding, is unable to restore on reflection their original speed: energy is passed through the boundary from inside outwards; and obviously the work done, or product of the distance

traversed into the resisting pressure, equals the *loss* of energy from the enclosure.

The more slowly the piston is moved the greater must be the resisting pressure and the greater is the transfer of energy. The limit to which this tends, the maximum amount of energy which could be extracted by moving the piston with infinite slowness to the extreme end of its possible motion; is the 'potential energy' of the enclosure: reckoned as negative in this case.

We can imagine the operation effected without our doing work as agents, if we suppose the piston's release and its attachment to an infinite outside mass to be effected by the shifting of a massless link.

As pointed out above; the pressure on the piston is, in our particular instance, always equal to twice the energy contained at the time in the envelope, divided by v; as the length of the cylinder measures its volume.

So $$-\Delta E = \frac{F+F+\Delta F}{2}\Delta v,$$

or at the limit, $$F_v = -\frac{dE_v}{dv} = \frac{2E_v}{v}; \text{ giving } E_v = \frac{K}{v^2}.$$

If the piston can move to infinity we have, potential energy

$$P = -\int_1^\infty F dv = -2\int_1^\infty \frac{E}{v}dv = \int_1^\infty \frac{-2K}{v^3}dv = -K = -E_1,$$

so that the sum of the kinetic and potential energies in this case was zero: as of course it must have been; for, during the infinite process of extension, every particle has had time to give up its whole energy to the yielding boundary.

If the pressure on the outside of the piston were greater than on the inside, the system,—consisting of the moving particles inside and the massless boundary,—would possess positive potential energy.

If the pressure inside is less than the outside pressure *while* the boundary is moving, we are not availing ourselves of the whole potentiality the arrangement possessed of receiving energy. As before, we may suppose an infinite mass attached

to the piston—but this time on its inner side;—that is, suppose the boundary moves inwards with infinite slowness under a balanced pressure.

If F is the pressure, active on the outside of the piston, and the liberty of motion is down to half volume; whether F is constant or not, so long as it is the active though 'balanced' pressure; we have,—

$$P_1 = -\int_1^{\frac{1}{2}} F dv; \text{ and at any stage } P_v = -\int_v^{\frac{1}{2}} F dv,$$

and $P_{\frac{1}{2}} = 0$; and the energy entering between v and $\frac{1}{2}$ is obviously $\int_{\frac{1}{2}}^{v} F dv$, since F is essentially positive.

$$\therefore\ E_v = E_{\frac{1}{2}} - \int_{\frac{1}{2}}^{v} F dv = E_{\frac{1}{2}} + \int_v^{\frac{1}{2}} F dv = E_{\frac{1}{2}} - P_v; \ \therefore\ E_v + P_v = E_{\frac{1}{2}}.$$

Or the sum of the potential and kinetic energies is as before constant, and here equal to $E_{\frac{1}{2}}$; the kinetic energy in the enclosure when the boundary has reached its extreme position, as before.

Thus we see that, in all cases, $-\dfrac{dE}{dv} = \dfrac{dP}{dv} = F$.............(21)

That direction being considered positive which is measured towards less constraint; and F, the equal pressure on both sides of the piston, being essentially positive; we see that $\dfrac{dE}{dv}$ is always negative; that $\dfrac{dP}{dv}$ is always positive; that each of these gives the arithmetical magnitude of F; and that the sign of P is opposite to that of the *acting* force F: but there is nothing here to determine what the sign of either P or acting F actually is.

To determine this important point we must consider the change of momentum at the boundary. Thus, in the case of two spheres in sequence direct, which we first considered; symmetry shewed us at once that the change of momentum in the fluid at the front of one sphere was identical with that at the back of the other; and therefore it was impossible for these pressures to accelerate both spheres or retard both; and as all resultant action lay along b, no energy could become

available unless b varied. That is, there was no source of energy to be applied to the 'outside' of our piston and no other 'piston' than that represented by b. Therefore the potential energy was necessarily negative, and acting F therefore positive, or acting in the direction of less constraint; which, as we had seen, was towards the other sphere.

In all the cases with which we are concerned, a simple way to decide the sign of the active pressure is to examine the form of the lines of flow. In this connection it may be remarked that the position of a line of flow is one of elastic equilibrium under the lateral pressure due to the acceleration on each side.

The discharge passing through any tube of flow is of course constant throughout the length of the channel. If the ends of the tube are fixed the equality between the discharge arriving at and leaving the areas covered by the ends, remains unaltered: but any deformation means greater pressure where there is greater curvature, *i.e* greater change of momentum. The ends of a line of flow on a moving sphere are fixed; for a certain definite discharge must reach or leave a certain element of its surface so long as it maintains a given velocity through a continuous fluid. If the line of flow be deformed, the greater pressure is at that end where the curvature is more increased: due allowance being made for any change in velocity.

If the ends of the lines of flow are not fixed, they will follow the lines of least resistance; so that an obstruction near one side of a sphere upon which movable lines of flow end or rise, will crowd them together on the opposite side; the greater momentum will approach or leave by the wider or more open path.

The ends of the lines of flow on a sink are not fixed: fluid is driven towards the centre of the sink from afar; but it is free to choose the easiest path. Anything, therefore, which detrudes the lines of flow of a sink,—whether that obstruction be some other lines of flow not ending on the sink, or a fixed boundary to which such other lines may be equivalent,—increases the pressure on the further side.

When stating that the sum of the potential and kinetic energies in any system is constant; no exception was made for the case of so-called non-conservative systems: the omission was intentional. It may be convenient on occasion to distinguish between conservative and non-conservative systems; but the difference is one of detail, not of principle. A non-conservative system is simply one in which there are a number of movable boundaries of which we cannot, or are not willing to take account. A fluid, for instance, which is not frictionless, simply involves the existence of a myriad 'pistons' of molecular dimensions.

There is, of course, a certain objection to the application of a proper name like Potential Energy to an integral function which has no independent existence, inasmuch as the use of such a term is apt to induce in the mind a belief in the concrete existence of what is but an abstract idea: kinetic energy is a real entity; potential energy is not. In this case, however, great convenience overweighs the risk of misapprehension. How great this convenience is, can best be illustrated by a simple instance.

Suppose we wish to determine the general form of the moon on the supposition that; either matter is now fairly free to move upon its surface, or the distance of the moon had something like its present value when last such freedom did exist.

Let K = constant of attraction; ρ = geocentric distance of the moon's centre; a = semiaxis major of the moon's form; b = semiaxis minor; m = earth's mass; m' = moon's attractive mass.

A unit mass on the moon's surface, originally at the extremity of a minor axis b and therefore distant almost exactly ρ from the earth's centre; if moved to the further extremity of a would thereby gain potential energy,

$$Km\int_{\rho}^{\rho+a}\frac{dr}{r^2}+\frac{2}{5}Km'\int_{b}^{a}\frac{dr}{r^2};$$

$\frac{2}{5}$ being the coefficient of correction which enables us to treat the moon's attractive mass as if concentrated at its centre.

A second unit mass at the extremity of b, if moved to the nearer extremity of a, would similarly 'gain' potential energy,

$$Km\int_{\rho}^{\rho-a}\frac{dr}{r^2}+\frac{2}{5}Km'\int_{b}^{a}\frac{dr}{r^2}.$$

If two such units have no tendency to move either thus or in the opposite direction, the sum of the above changes in potential energy must be zero, or,

$$\frac{a-b}{b}=\frac{5}{2}\frac{m}{m'}\frac{a^3}{\rho(\rho^2-a^2)}:$$

in which, since $\frac{a-b}{b}$ is a very small fraction, we may take for a, on the right-hand side of the equation, the semidiameter of the moon as seen by us. This gives $\frac{a-b}{b}=\cdot000018$; representing a divergence from the circular form far too minute to be appreciated by the eye. The value of $a-b$, though only 103 feet, would still be amply sufficient to retain the same side of the moon directed to the earth *after* all surplus angular momentum had been destroyed by such tremendous tidal effort on a globe while it was still comparatively viscous.

That we were able by the use of an approximately constant coefficient (for the coefficient $\frac{2}{5}$ is not more than a very close approximation to the true coefficient) to evaluate potential on the assumption that the moon's mass was concentrated at its centre, is a secondary but striking illustration of the extreme convenience of that function of energy and position; which will be fully appreciated by anyone who has attempted to calculate the attraction of an ellipsoid on an exterior particle.

Taking potential energy as we have defined it; the loss of potential energy in an attracted particle due to its transfer from infinity to a given point, may be called the Veetal of that point.

The veetal at the 'pole,' or extremity of the long axis, of an attracting prolate spheroid is,

$$V_a=\frac{KM}{a}\left\{1+\frac{1}{5}e^2+\frac{3}{35}e^4,\text{ etc.}\right\};$$

and at the 'equator' is,

$$V_b = \frac{KM}{a}\left\{1+\frac{2}{5}e^2+\frac{9}{35}e^4, \text{ etc.}\right\}.$$

So the difference in potential between pole and equator is,

$$\frac{KM}{a}\left\{\frac{e^2}{5}, \text{ etc}\right\};$$

in which $e^2 = \frac{a^2-b^2}{a^2}$: and, when a is very nearly equal to b, we see that $e^2 = \frac{a+b}{a} \cdot \frac{a-b}{a}$ is very nearly equal to $2\frac{a-b}{b}$. The difference of potential on the assumption that the mass of the ellipsoid is all concentrated at its centre would, of course, be $\frac{KM}{a} \cdot \frac{a-b}{b}$: hence our coefficient.

IV

SOURCES AND SINKS

NEXT in order of simplicity to a spherical shell would appear a source or sink in a fluid, as representing some particle or local difference: indeed, from one point of view this is a more elementary and simple conception, inasmuch as it might seem to necessitate the existence of nothing beyond the fluid itself; but correctly viewed the two suppositions are identical in order of simplicity; one being the negative of the other.

A source or sink implies an aperture in a boundary. A spherical shell is a boundary separating a small sphere from the rest of space; the negative of this may be imagined to bound the rest of space exclusive of the 'real' sphere: or it may be viewed as an aperture taking the place of an obstruction.

We may, if we choose, picture a source and a sink as apertures on opposite sides of our universe, in by one of which and out by the other flows an ether stream due to the motion of our universe in four-dimensional space; or if we prefer not to suppose anything unobvious to our perceptive senses, we may,—though with really less probability of truth,—picture a source as a condensed particle of fluid from which fluid is evaporating, and a sink as a similar particle upon which fluid is being precipitated.

Our present object, however, is not to choose between these or any other possible hypotheses: we simply have to examine the necessary consequence of the two existences we have assumed,—a pervading medium and a localisation: leaving all question as to the possible nature of each, entirely open.

We will therefore look upon sources and sinks simply as

spherical surfaces, from or towards the convex surfaces of which our medium is being directly projected.

Adopting this abstract idea let us consider a single spherical sink of radius a, and discharge $-4\pi K$; alone and stationary in an infinite fluid. If we call V the radial velocity at the surface of the sphere, then $K=Va^2$: and generally, for all points outside the sphere the velocity is radial, and equals $\dfrac{-K}{r^2}=\dfrac{-Va^2}{r^2}$.

The volume of an element having a given velocity is $4\pi r^2 dr$; half the square of the velocity is $\dfrac{V^2a^4}{2r^4}$; so,—taking as before unity for the fluid's density,—the total energy is—

$$2V^2a^4\pi\int_a^\infty \frac{dr}{r^2}=2V^2a^3\pi, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

or six times that of a sphere moving with velocity V; but only $\frac{3}{2}$ that due to a moving sphere which has the same total discharge, front and back added.

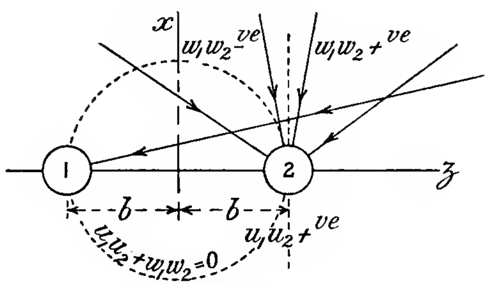

FIG. 5.

Now let us consider two of these sinks; one at $z=-b$, and the second at $z=+b$. Everything being symmetrical about the axis of z, we may use cylindrical coordinates; and our component velocities will be,—

$$u_1=\frac{-Va^2x}{\{x^2+(z+b)^2\}^{\frac{3}{2}}};\quad u_2=\frac{-Va^2x}{\{x^2+(z-b)^2\}^{\frac{3}{2}}}.$$

$$w_1=\frac{-Va^2(z+b)}{\{x^2+(z+b)^2\}^{\frac{3}{3}}};\quad w_2=\frac{-Va^2(z-b)}{\{x^2+(z-b)^2\}^{\frac{3}{2}}}.$$

Now our spherical sink may or may not be an obstruction impervious to the lines of flow of another sink. If it forms no obstruction, then the resultant velocity at any point in space will truly be $\{(u_1+u_2)^2+(w_1+w_2)^2\}^{\frac{1}{2}}$: and if it do constitute an obstruction we can subsequently allow for the obstructive effect as before.

The 'additional' energy, therefore, which is due to the coexistence of the two sinks, is,—

$$V^2a^4\pi\iint 2x\frac{(x^2+z^2-b^2)dx\,dz}{\{x^4+2x^2(z^2+b^2)+(z^2-b^2)^2\}^{\frac{3}{2}}}.$$

Putting $m=x^2+z^2+b^2$; giving $\dfrac{dx}{dm}=\dfrac{1}{2x}$; we obtain,—

$$V^2a^4\pi\iint\frac{m-2b^2}{\{m^2-4b^2z^2\}^{\frac{3}{2}}}\,dm\,dz=V^2a^4\pi\int\frac{m-2z^2}{2z^2\{m^2-4b^2z^2\}^{\frac{1}{2}}}\,dz.$$

When $x=\infty$, $m=\infty$, and when $x^2=a^2-(z-b)^2$, $m=a^2+2bz$. Therefore round the sphere and between $z=b-a$ and $z=b+a$, we have,—

$$V^2a^4\pi\int_{b-a}^{b+a}dz\left\{\frac{1}{2z^2}+\frac{2z^2-2bz-a^2}{2z^2a\{a^2+4bz\}^{\frac{1}{2}}}\right\}$$

$$=V^2a^4\pi\left[\frac{-1}{2z}+\frac{(a^2+4bz)^{\frac{1}{2}}}{2ab}+\frac{(a^2+4bz)^{\frac{1}{2}}}{2az}\right]_{b-a}^{b+a}$$

$$=V^2a^4\pi\left\{\frac{-1}{2(b+a)}+\frac{1}{2(b-a)}\pm\frac{2b+a}{2ab}\mp\frac{2b-a}{2ab}\right.$$

$$\left.\pm\frac{2b+a}{2a(b+a)}\mp\frac{2b-a}{2a(b-a)}\right\}$$

$$=V^2a^4\pi\frac{1}{b+a}\cdot\frac{a}{b}\cdot \qquad\qquad (23)$$

if we take the upper signs; and it must be right to do this, for $\sqrt{a^2+4bz}$ is simply the distance from the centre of sink 1 to the lower limit of our integration to x, an essentially positive quantity. That the additional energy must be positive is readily seen by reference to fig. 5. Here, between $z=b-a$ and $z=b+a$, u_2 is evidently powerful and exactly equal at corresponding points to right and left of $z=b$: while u_1 conspires with it on both sides. w_2 is also symmetrical and small except in the cusps; w_1 is additive on one side and subtractive

on the other; for it retains the same sign on crossing $z=b$, while w_2 changes sign: and the value of w_1 is very slightly in excess on one side over its value on the other,—evidently the total must be positive.

For the rest of space to the right of the origin; since when $x=0$, $m=z^2+b^2$, we have—

$$V^2a^4\pi\int dz\left\{\frac{1}{2z^2}\mp\frac{b^2-z^2}{2z^2(b^2-z^2)}\right\}$$
$$=V^2a^4\pi\int\left(0,\text{ or }\frac{1}{z^2}\right)dz.\ \ldots\ldots\ldots\ldots(24)$$

Concerning the sign of the second term in this case: our original function is zero when $x^2+z^2-b^2=0$, or the orthogonal surface is a sphere about the origin whose radius is b (fig. 5). Here we see that u_1u_2 is everywhere positive; that w_1w_2 is negative between $z=-b$ and $z=+b$, everywhere else positive: $u_1u_2+w_1w_2$ is therefore positive beyond $z=b$; and being everywhere finite it cannot change sign except on passing through zero. Our function is therefore positive outside the dotted circle and negative inside.

It follows—

(i) That our integral to x may be zero between $z=-b$ and $z=+b$; but it must be positive beyond.

(ii) At $z=0$, being of the form specified, it cannot be other than zero, else its integral to z would be infinite.

(iii) b^2-z^2 in the numerator,—which has not been squared—changes sign as z passes b.

It follows that the integral to x between $z=0$ and $z=b-a$ is zero, and between $b+a$ and ∞ it is $\dfrac{V^2a^4\pi}{z^2}$.

So our total 'additional' energy over half space is,—

$$V^2a^4\pi\frac{1}{b+a}\cdot\frac{a}{b}+V^2a^4\pi\int_{b+a}^{\infty}\frac{dz}{z^2}=\frac{2V^2a^4\pi}{d},\ \ldots\ldots\ldots(25)$$

or the total energy in the fluid over half space when two sinks coexist, is,—

$$\frac{E}{2}=\frac{2\pi K^2}{a}+\frac{2\pi K^2}{d}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

Giving
$$\frac{dE}{dd}=-\frac{4\pi K^2}{d^2}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

It is noteworthy that this last does not involve a if we suppose the discharge fixed: so that, whether the spherical sink we have imagined obstructs motion in the fluid or not, the force must vary inversely as the square of the distance so long as d does not approach atomic dimensions.

Some lines of flow for a pair of sinks are shewn in fig. 6; their spacing, however, bears no reference to the density of the discharge. With or without reference to this figure, it is obvious that the lines of flow of each sink detrude those of the other from the nearer towards the further side; where, therefore, is the greater pressure. F, then, is a force of attraction; already shewn to vary inversely as the square of the distance.

If we substitute a pair of sources for the sinks, exactly the same results are obtained; for changing everywhere the signs of all the velocities changes neither their products nor their squares.

In both cases, it will be noticed, w_1+w_2 is zero over the plane $z=0$. It therefore follows that either individual of a pair,—so far as the other is concerned,—has exactly the effect of a plane fixed boundary $z=0$.

It is very easy to experimentally prove the existence of attraction between a source, or a sink, and an adjacent plane boundary, in such a fluid as water, by simply suspending any symmetrical substitute for our ideal source or sink in a moderately large vessel, and fixing a sheet of glass in its neighbourhood.

It appears natural enough to see a sink drawn towards a neighbouring plane; but, at first sight, there is something surprising about the sudden rush towards an obstructing plane made by an aperture from which water is rushing against the obstruction. The surprise is, of course, but momentary; more momentum is being delivered from the free side than from that which is partly obstructed.

If, therefore, we can shew that the ultimate particles of our universe are either sources or sinks, the phenomenon of gravitation is adequately explained: and that not in the sense in which the word explanation is so frequently used,—the bringing of one phenomenon under the scope of some other

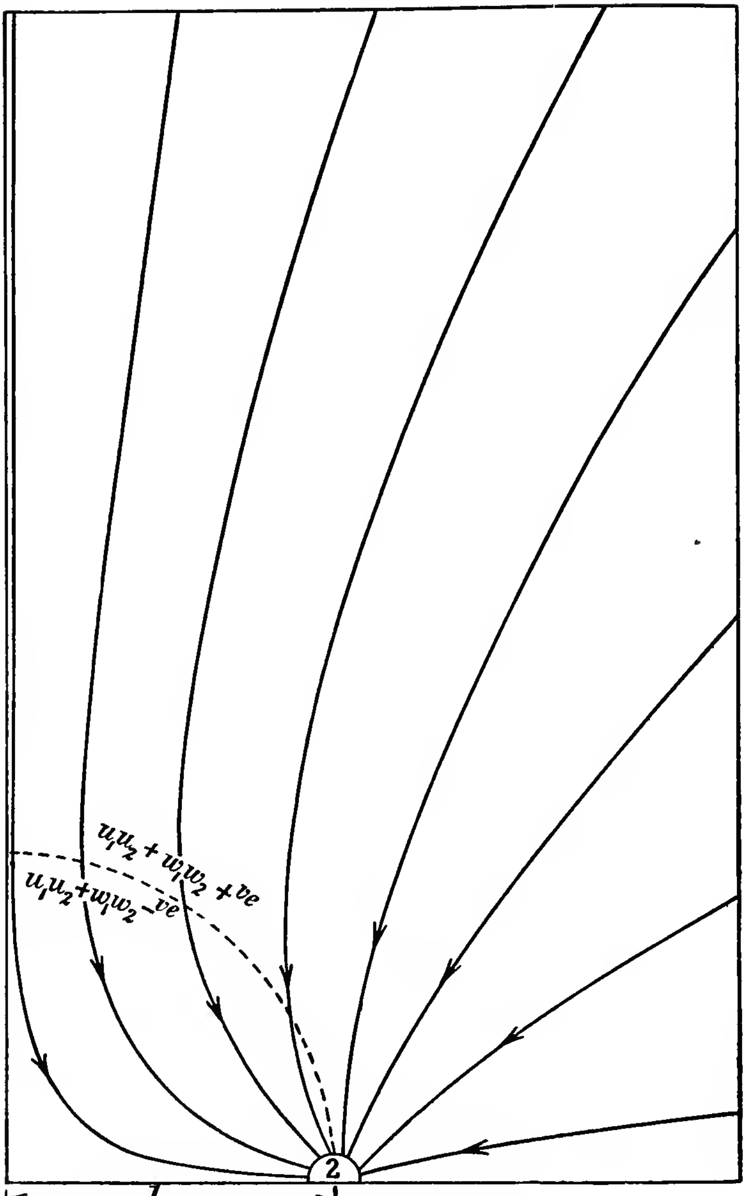

FIG. 6.—TWO SPHERICAL SINKS.

known 'law of nature,' a process which no more truly explains the phenomenon than giving a Latin name to a fungus explains its existence,—but the reduction of a more or less complicated expression to its essential arithmetical identity.

A sink and a source of course repel each other. Reversing everywhere the sign of one of the compounding velocities in our product, reverses the sign of the function we had to sum; and our 'additional' energy is negative.

Now if our sinks and sources do present obstruction to the flow of fluid,—and any physical conception of a sink appears to require this,—we must allow for the effect: and this, of course, can be done as before.

In passing, we may form some idea concerning what this correction amounts to when the sinks are, say ten radii apart, by reference to fig. 7, which shews the lines of flow for a pair of two-dimensional sinks *and* the lines of flow for a pair of circles in sequence reversed, in a plane. The corresponding curve series for the three-dimensional problem are, at a glance, scarcely distinguishable from those of the two-dimensional case, while the latter are much more readily drawn. Remembering that the velocities in the moving spheres' dotted lines of flow are, at their maximum, only one hundredth of the velocities in the sinks' lines of flow; we see that when the two series are compounded, the sinks' lines of flow will simply experience a minutely increased detrusion.

In evaluating this correction for impermeability we shall require the 'additional energy' due to the coexistence of a sink and a moving sphere.

Since our moving sphere in this case is the 'correctional' sphere whose motional effect is to be substituted for the obstruction offered by the sink, the radii of the sink and sphere are alike; but we must distinguish between their velocities.

A moving sphere at $z=-b$ gives,—

$$u_3=\frac{V'a^3}{2}\cdot\frac{3x(z+b)}{(x^2+2bz+b^2+z^2)^{\frac{5}{2}}},$$

$$w_3=\frac{V'a^3}{2}\cdot\frac{-x^2+2z^2+4bz+2b^2}{(x^2+2bz+b^2+z^2)^{\frac{5}{2}}}.$$

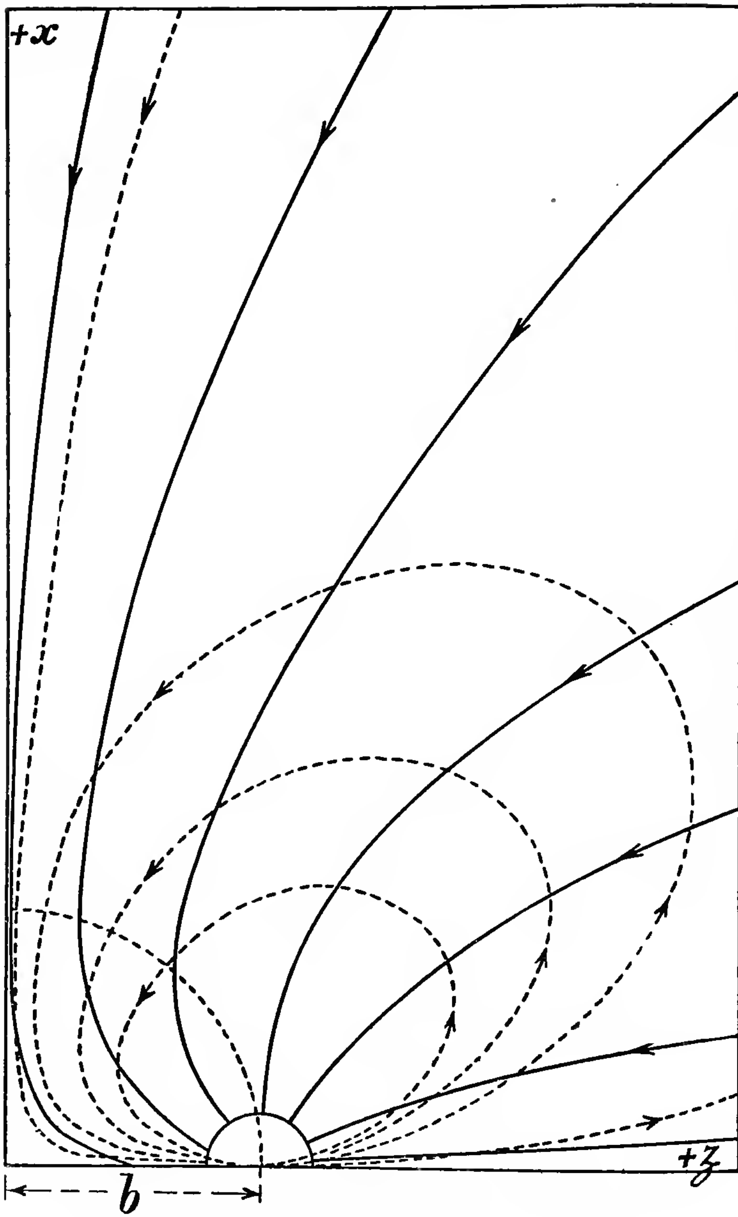

FIG. 7.—A PAIR OF TWO-DIMENSIONAL SINKS AND A PAIR OF CIRCLES IN SEQUENCE REVERSED.

A sink at $z = +b$ gives,—

$$u_2 = -Va^2 \cdot \frac{x}{(x^2+z^2-2bz+b^2)^{\frac{3}{2}}},$$

$$w_2 = -Va^2 \frac{z-b}{(x^2+z^2-2bz+b^2)^{\frac{3}{2}}},$$

the additional energy we seek being,—

$$2\pi \iint (u_3u_2 + w_3w_2) x\,dx\,dz.$$

Now the denominators of the fractions concerned, when these fractions have been multiplied together and brought to a common denominator, contain no odd powers of z under the root; and are therefore unaltered in magnitude by change in sign of z; and being powers of the radii vectores and their products, are essentially positive. All the terms then found in the numerator to have an odd power of z, will, therefore, cancel when summed over all space. These, therefore, we may omit entirely; and all we have to integrate reduces to,—

$$2VV'a^5\pi b \iint \frac{-2x^4 + x^2(-z^2-b^2) + (z^2-b^2)^2}{\{x^4+2x^2(z^2+b^2)+(z^2-b^2)^2\}^{\frac{5}{2}}} x\,dx\,dz. \quad\text{.....(28)}$$

Put $m = x^2+z^2+b^2$, giving $\frac{dx}{dm} = \frac{1}{2x}$, and we have,—

$$VV'a^5\pi b \int dz \int \frac{-2m^2 + m(3z^2+3b^2) - 4b^2z^2}{\{m^2-4b^2z^2\}^{\frac{5}{2}}} dm$$

$$= VV'a^5\pi b \int \frac{m - z^2 - b^2}{\{m^2-4b^2z^2\}^{\frac{3}{2}}} dz;$$

or translated back into x for possible reference,

$$VV'a^5\pi b \int \frac{x^2}{\{x^4+2x^2(z^2+b^2)+(z^2-b^2)^2\}^{\frac{3}{2}}} dz. \quad\text{.........(29)}$$

The value of this integral to x, between $x=0$ and $x=\infty$, is zero; so all we have to integrate over is the space between $z = b+a$, and $b-a$.

When $x^2=a^2-(z-b)^2$, $m=a^2+2bz$; and we have,—

$$VV'a^5\pi b\int\left\{0-\frac{a^2-z^2-b^2+2bz}{a^3\{a^2+4bz\}^{\frac{3}{2}}}\right\}dz$$

$$=VV'a^2\pi\left\{\frac{2b^2z^2+z(-2ba^2-12b^3)-a^4-6b^4}{12b^2\{a^2+4bz\}^{\frac{1}{2}}}\right\}.$$

This taken between $z=b-a$, and $b+a$, gives $\dfrac{-VV'a^5\pi}{6b^2}$; so the additional energy over all space for a sphere moving with velocity V' towards a sink of discharge $4\pi a^2V$, is,—

$$\frac{-4VV'a^5\pi}{3d^2}\text{.........................}(30)$$

It is desirable to satisfy ourselves that no error is involved in our assertion that the integral to x of the function which we summed, was zero when taken between $x=0$, and $x=\infty$. What we had first to integrate, shewn in (28), is of course zero when $x=0$ because the annular element of space then concerned is zero; but after dividing out the x, our space scalar is zero when

$$x^4+x^2\frac{z^2+b^2}{2}=\frac{(z^2-b^2)^2}{2}$$

or when $$x^2=\frac{\sqrt{9z^4+9b^4-14z^2b^2}-z^2-b^2}{4}.$$

And this always gives a real positive value for x, so long as z does not equal b; and if z does equal b, we do not integrate from $x=0$, since that would be the centre of the sphere or sink: and, because the coefficients of x in our function are both negative, it follows that below the above value our function is positive, and above negative; varying continuously from its extreme positive value when $x=0$, and its extreme negative value when $x=\infty$, in each direction to this zero value. Since, then, our integral to x is zero at $x=0$ and at $x=\infty$, these two branches truly cancel.

We are not now concerned with the force, if any, which this represents between the moving sphere and the sink; what is required being the sum of the products of the velocity components for use in correcting our first approxima-

tion to the action between two coexisting sinks. This sum having been obtained for the case in which the correctional sphere and the sink are distant $2b$ from each other, all that we still want is the similar quantity for the case of sink and sphere coinciding in position.

Here we have,

$$u_4u_2 = -\frac{VV'a^5}{2}\,\frac{3x^2z}{(x^2+z^2)^4}, \quad \text{and} \quad w_4w_2 = -\frac{VV'a^5}{2}\,\frac{2z^3-zx^2}{(x^2+z^2)^4};$$

the origin being at the centres.

Now the denominators are simply r^8 and necessarily positive. The numerators contain nothing but terms comprising even powers of x and odd powers of z: and since the radius of the sink is equal to that of the sphere, it follows that for every element of space with a given velocity product to the right, there is an equal element with a velocity of equal magnitude but of opposite sign, on the left. The sum over all space of this product therefore vanishes; and we have zero for the 'additional' energy of coincident sink and moving sphere.

We are now in a position to correct (26) for impermeability. If $nV = V_1$ be the effective velocity against a sphere at $z = -b$, due to a sink, the positive value of whose discharge is $4\pi a^2 V$, at $z = +b$; then $-V_1$ is the velocity of sphere 3, which is therefore moving away from the sink 2; and $+V_1$ is that of sphere 4, which is therefore moving away from sink 1: and if mV_1 be the velocity effective against a sphere at $z = -b$, due to a moving sphere, the velocity of which is V_1, at $z = +b$; then the velocity of sphere 5 is $-V_1m$; and of 6 is $+V_1m$; or in each case away from the distant sink: also $u_5 = mu_3$, and $u_6 = mu_4$.

Therefore for the total energy we have,—

$$\int\frac{u_1^2+u_2^2}{2} = 2V^2a^3\pi \times 2; \qquad \int u_1u_2 = \frac{4V^2a^4\pi}{d}.$$

$$\int\frac{u_3^2+u_4^2}{2} = \frac{V^2a^3\pi}{3} \times 2n^2; \qquad \int u_1u_4 + u_2u_3 = \frac{4V^2a^5\pi}{3d^2} \times 2n.$$

$$\int u_1u_3 + u_2u_4 + u_1u_5 + u_2u_6 = 0; \qquad \int u_1u_6 + u_2u_5 = \quad ,, \quad \times 2m\,.\,n.$$

$$\int u_3u_5 + u_4u_6 = \frac{V^2a^3\pi}{3} \times 4n^2m; \qquad \int u_3u_4 = \frac{2V^2a^6\pi}{3d^3} \times n^2.$$

Or $$\frac{E}{2}=\frac{V^2a^3\pi}{3}\{6+n^2+2n^2m\}$$
$$+\frac{V^2a^3\pi}{3}e\{6+4ne+4nme+n^2e^2\}. \quad\ldots\ldots\ldots\ldots(31)$$

If we take the central for the effective velocities, $n=e^2$; and $m=e^3$; this gives,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\{6+6e+5e^4+7e^7\}. \quad\ldots\ldots\ldots\ldots(32)$$

To obtain a better value for n, we may calculate the mean velocity parallel to z of a sink at the origin, over the surface $x^2=a^2-(d-z)^2$.

Velocity $$w=\frac{-za^2V}{(x^2+z^2)^{\frac{3}{2}}};$$

over above surface,—since $\frac{dx}{dz}=\frac{d-z}{x}$; gives discharge

$$-Va^22\pi\int_0^{a\,;\,a}\frac{zx}{(x^2+z^2)^{\frac{3}{2}}}dx=-Va^22\pi\int_{d-a\,;\,d+a}^{d}\frac{dz-z^2}{(a^2-d^2+2dz)^{\frac{3}{2}}}dz$$
$$=-Va^22\pi\{2\phi(d)-\phi(d-a)-\phi(d+a)\};$$

where $$\phi(z)=\frac{-z^2}{3d\{a^2-d^2+2dz\}^{\frac{1}{2}}}+\frac{z(2a^2+d^2)}{3d^2\{\ ,,\ \}^{\frac{1}{2}}}+\frac{2a^4-d^2a^2-d^4}{3d^3\{\ ,,\ \}^{\frac{1}{2}}};$$

$$\phi(d-a)=\frac{-2a^3}{3d^3}-\frac{1}{3};\quad \phi(d+a)=\frac{2a^3}{3d^3}-\frac{1}{3};$$

$$\phi(d)=\frac{2a^4+a^2d^2-d^4}{3d^3\{d^2+a^2\}^{\frac{1}{2}}}=\frac{-1}{3}+\frac{e^2}{2}+\frac{3e^4}{8}-\frac{5e^6}{48}.$$

Giving mean velocity,

$$-V\left\{e^2+\frac{3}{4}e^4-\frac{5}{24}e^6\right\}.\quad\ldots\ldots\ldots\ldots(33)$$

Using this value for $-nV$; and our formerly obtained value for m; *i.e.* $e^3+\frac{3}{2}e^5-\frac{5}{8}e^7$; we obtain,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left\{6+6e+5e^4+\frac{9}{2}e^6,\text{ etc.}\right\}.\quad\ldots\ldots\ldots\ldots(34)$$

There remains the correction for 'enclosed' energy.

The energy, due to a sink at the origin, enclosed within the surface $x^2=a^2-(z-d)^2$ is, integrated over a plane perpendicular to z,—

$$V^2a^4\pi\int_0^{x^2=a^2-(z-d)^2}\frac{x}{(x^2+z^2)^2}dx=V^2a^4\pi\left\{\frac{1}{2z^2}-\frac{1}{2(a^2-d^2+2dz)}\right\}.$$

The integral of which to z is,

$$V^2a^4\pi\left\{\frac{-1}{2z}-\frac{1}{4d}\log(a^2-d^2+2dz)\right\}.$$

The value of which between $z=b-a$ and $z=b+a$ is,—

$$V^2a^4\pi\left\{\frac{a}{d^2-a^2}-\frac{1}{2d}\log\frac{d+a}{d-a}\right\}=\frac{V^2a^3\pi}{3}\left(2e^4+\frac{12}{5}e^6+\frac{18}{7}e^8\right). \quad (35)$$

The energy, due to one correctional sphere, enclosed in the other first correctional sphere; as before shewn, would contain as its lowest term e^6 multiplied into the square of the sphere's velocity. The sphere's velocity here being e^2V; the lowest term would be e^{10}. The whole deduction to be made on this account is therefore given by (35).

Thus we obtain for a pair of three-dimensional sinks,

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left\{6+6e+3e^4+\frac{21}{10}e^6+7e^7\right\}, \quad\ldots\ldots\ldots\ldots(36)$$

$$\frac{dE}{dd}=-\frac{V^2a^4\pi}{d^2}\left\{4+8e^3+\frac{42}{5}e^5+\frac{98}{3}e^6\right\}.\quad\ldots\ldots\ldots\ldots(37)$$

From which the last two terms should be discarded, as being in error.

From what precedes (33), we see that the real discharge on the near side of correctional sphere 4 is $-\frac{2}{3}e^3\times 2\pi a^2V$ besides the amount allowed for in our correction when we used the 'mean velocity'; and that, on the far side, it is $+\frac{2}{3}e^3 2\pi a^2 V$ besides what is included in our mean discharge. These discharges from the sphere when stopped are in effect that of a part z sink of discharge $\frac{8}{3}e^3\pi a^2V$.

Now the neglected radial discharge through this correctional sphere is; origin at centre of sink 1,—

$$-Va^2 2\pi\int_{d-a}^{d+a}\frac{x^2}{(x^2+z^2)^{\frac{3}{2}}}dz = -2\pi Va^2\int_{d-a}^{d+a}\frac{a^2-z^2-d^2+2dz}{(a^2-d^2+2dz)^{\frac{3}{2}}}dz$$

$$=2Va^2\pi\left[\frac{z^2}{3d\{\ ,,\ \}^{\frac{1}{2}}}+\frac{z(-2a^2-4d^2)}{3d^2\{\ ,,\ \}^{\frac{1}{2}}}+\frac{-2a^4+d^2a^2+d^4}{3d^3\{a^2-d^2+2dz\}^{\frac{1}{2}}}\right]_{d-a}^{d+a}$$

$$=-\frac{8Va^2\pi}{3}e^3.$$

The stoppage of this discharge onto the sphere is equivalent to an x part source of discharge $\frac{8}{3}e^3\pi a^2 V$.

Thus we have a source-sink which we are neglecting here just as we neglected a sink-source in the case of spheres in sequence; but here we note that our solitary energy ignored is of the order e^6: and this source-sink has a complete sink at 1 of velocity V with which to compound; and, as we have seen, moving spheres compound with sinks to produce additional energy of order $VV'e^2$: so, as in this case $V'=Ve^3$, our neglected additional energy will be of the order e^5. We must therefore discard terms above e^4. The remainder is, however, amply sufficient: and quite as close an approximation as that which we considered sufficient in the case of moving spheres: or comparatively speaking a closer approximation; for then our dominant power was e^3, and we pursued correction to e^6. Here our dominant term is e, and we pursue correction to e^4.

V

TWO CIRCLES

FOR purposes of comparison it will be well to consider the two-dimensional problem. Conceive a circle moving in its own plane: or, if we prefer this, an infinite circular cylinder moving in a direction perpendicular to its own axis in a three-dimensional medium.

The 'velocity potential' being $\phi = -\dfrac{Va^2 \cos\theta}{r}$; we have for the radial velocity in the medium $\rho = \dfrac{Va^2 \cos\theta}{r^2}$, and for the transversal $\tau = \dfrac{Va^2 \sin\theta}{r^2}$. The energy in the fluid is therefore $\dfrac{V^2 a^4}{2r^4}$ at every point; or over all space,—since our elemental area is $r d\theta dr$, we have,—

$$2V^2 a^4 \int_a^\infty dr \int_0^{\frac{\pi}{2}} \frac{d\theta}{r^3} = \frac{V^2 a^2 \pi}{2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(38)$$

or the energy of a circle of the same density as the medium, moving with a velocity V.

It is noteworthy that the absolute speed, the momentum and the energy at any point are independent of θ, being functions of the distance from the centre of the moving circle, only.

The equation to the lines of flow is $c = \dfrac{r}{\sin\theta}$; that of circles tangent to the axis of r or z along which the circle moves.

Let there be one circle, 1, at $z = -b$ and another, 2, at $z = +b$ each moving with velocity V along the axis of z.

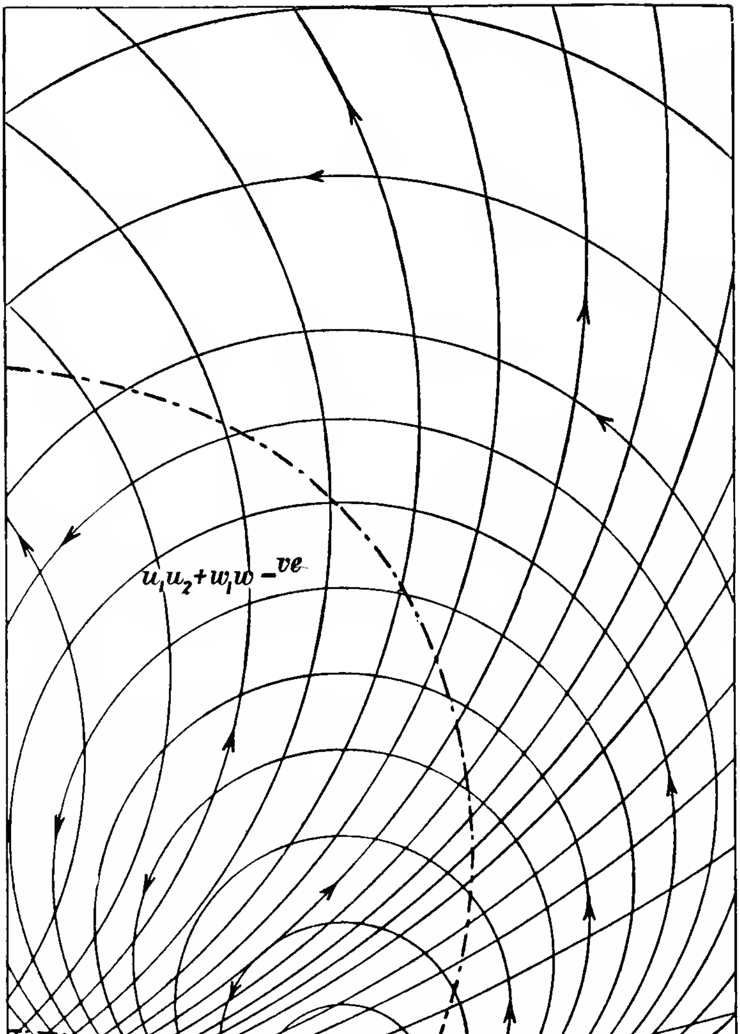

FIG. 8.—CIRCLES IN SEQUENCE: SOLITARY LINES OF FLOW.

We have,—

$$u_1 = Va^2 \frac{2xz+2xb}{(x^2+z^2+b^2+2bz)^2}, \quad u_2 = Va^2 \frac{2xz-2xb}{(x^2+z^2+b^2-2bz)^2},$$

$$w_1 = Va^2 \frac{z^2+b^2+2bz-x^2}{(x^2+z^2+b^2+2bz)^2}, \quad w_2 = Va^2 \frac{z^2+b^2-2bz-x^2}{(x^2+z^2+b^2-2bz)^2}.$$

Seeking, as before, the 'additional energy' on the assumption that each circle is permeable to the lines of flow due to the other; we have,—

$$\int \frac{u_1 u_2 + w_1 w_2}{V^2 a^4} dx = \int \frac{x^4 + 2x^2(z^2-3b^2) + (z^2-b^2)^2}{\{x^4 + 2x^2(z^2+b^2) + (z^2-b^2)^2\}^2} dx$$

$$= \frac{1}{4z^2}\left\{\frac{x}{x^2+(z+b)^2} + \frac{x}{x^2+(z-b)^2} + \frac{1}{z}\tan^{-1}\frac{x}{z+b} + \frac{1}{z}\tan^{-1}\frac{x}{z-b}\right\}. \tag{39}$$

The value of this above the circle 2, or between

$$x^2 = a^2 - (z-b)^2 \text{ and } x = \infty,$$

is,
$$\frac{1}{4z^2}\left\{\frac{-\sqrt{a^2-(z-b)^2}}{a^2+4bz} + \frac{-\sqrt{a^2-(z-b)^2}}{a^2}\right.$$
$$\left. + \frac{\pi}{z} - \frac{1}{z}\tan^{-1}\frac{\sqrt{a^2-(z-b)^2}}{z+b} - \frac{1}{z}\tan^{-1}\frac{\sqrt{a^2-(z-b)^2}}{z-b}\right\}. \quad \ldots(40)$$

The integral of which to z is,—

$$\frac{1}{8z^2}\left(\tan^{-1}\frac{\sqrt{a^2-(z-b)^2}}{z+b} + \tan^{-1}\frac{\sqrt{a^2-(z-b)^2}}{z-b}\right) + \frac{1}{2a^2}\tan^{-1}\left(\frac{2b-a}{2b+a}\right.$$
$$\left.\times\frac{\sqrt{a-(z-b)}}{\sqrt{a+(z-b)}}\right) - \frac{1}{4a^2}\cos^{-1}\frac{z-b}{a} + \frac{1}{4a^2}\left\{\frac{\sqrt{a^2-(z-b)^2}}{z}\right\} - \frac{\pi}{8z^2}. \tag{41}$$

This taken between $z=b-a$ and $z=b+a$, gives,—

$$\frac{1}{8(b+a)^2}\left(\tan^{-1}(+0)+\tan^{-1}(+0)\right) - \frac{1}{8(b-a)^2}\left(\tan^{-1}(+0)+\tan^{-1}(-0)\right)$$
$$+\frac{1}{2a^2}\tan^{-1}(+0) - \frac{1}{2a^2}\tan^{-1}(+\infty) - \frac{1}{4a^2}\cos^{-1}(+1)$$
$$+\frac{1}{4a^2}\cos^{-1}(-1) - \frac{\pi}{8(b+a)^2} + \frac{\pi}{8(b-a)^2}$$
$$= \frac{-\pi}{8(b-a)^2} - \frac{\pi}{4a^2} + \frac{\pi}{4a^2} - \frac{\pi}{8(b+a)^2} + \frac{\pi}{8(b-a)^2} = \frac{-\pi}{8(b+a)^2}. \quad \ldots(42)$$

By $\tan^{-1}(+0)$ we mean that the function in question when approaching the limit zero, on the side with which we are concerned, was positive.

For the rest of space to the right of the origin we require our original integral to x (39) taken between $x=0$ and $x=\infty$; which gives,—

$$\frac{1}{4z^2}\left\{0-0+0-0+\frac{1}{z}\left(\tan^{-1}\frac{\infty}{z+b}+\tan^{-1}\frac{\infty}{z-b}\right)\right.$$
$$\left.-\frac{1}{z}\left(\tan^{-1}\frac{0}{z+b}+\tan^{-1}\frac{0}{z-b}\right)\right\}.$$

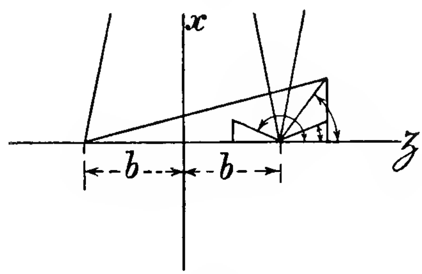

FIG. 8a.

Now we can see from fig. 8a that, throughout the positive quadrant in x and z over which we are integrating, $\tan^{-1}\frac{\infty}{z+b}$ and $\tan^{-1}\frac{\infty}{z-b}$ are each equal to $\frac{\pi}{2}$: and that $\tan^{-1}\frac{0}{z+b}=0$; but that $\tan^{-1}\frac{0}{z-b}$ is zero beyond $z=b$, but equals π between $z=0$ and $z=b$. Therefore, between the two circles our integral to x between 0 and ∞ is zero; while beyond circle 2 it is $\frac{\pi}{4z^3}$.

The rest of our additional energy, divided by V^2a^4, is therefore,—

$$\frac{\pi}{4}\int_{b+a}^{\infty}\frac{dz}{z^3}=\frac{\pi}{8(b+a)^2}.$$

Adding this to our previous result we find that the additional energy, under the assumption that each circle is permeable, is zero.

This interesting result is evidently due to the fact that both lines of flow and isoenergic lines, in the case of a solitary moving circle, are circles.

The case was, of course, quite different with moving spheres. It is true that in both cases the lines of flow about the equator of a circle or sphere are in opposition to those due to the further circle or sphere which pass through that region; or each is a negative constraint on the other: that is to say, a motion of the fluid equal to the difference between the motions required for each sphere or circle when alone, is enough for the requirements of both when they coexist. A great saving of energy thus results: as a saving in expenditure necessary for the working of two factories would result from their juxtaposition, if each utilized the waste products of the other. Exactly the reverse of this holds in the case of permeable circles or spheres about their 'poles': here the lines of flow conspire; and the constraint is positive. But in the case of two circles the speed at the pole is equal to the speed at the equator, and the volume or mass represented by an element of area on our diagram at the pole is the same as that represented by a similar area at the equator; while in the case of spheres, although the speed at the equator is only half that at the poles, the volume or mass of the fluid represented by an area $\Delta x \Delta z$ on the plane of our diagram at the equator, bears to the mass represented by a similar area at the poles, the ratio $\frac{4a}{\Delta x}+2$: which, of course, is infinite *at* the limit and still very great for all neighbouring regions. It follows that, in the case of two spheres, there is a balance of negative constraint, while in the two-dimensional problem it appears that the two effects exactly cancel.

As before, we can trace the curve where the additional energy is zero. This,—the orthogonal intersections of the solitary lines of flow,—is, for two circles in sequence,—

$$x^4+2x^2(z^2-3b^2)+(z^2-b^2)^2=0,$$

a curve similar in general form to that shewn by the section

of the corresponding three-dimensional surface; but in this case it resolves into

$$\{(x-b)^2+z^2-2b^2\}\{(x+b)^2+z^2-2b^2\}=0,$$

or two circles; centres at $x=\pm b$; and radii $b\sqrt{2}$: shewn by the dot and dash line in fig. 8.

This figure also shews the solitary lines of flow for two circles moving in the same direction along z.

The medium is at rest at the spots $z=0$, $x=\pm b$; and across the rest of the line $z=0$ it is moving parallel to z, with positive velocity between these points and negative velocity beyond them: fig. 9.

The fact that the energy in the medium is unaltered by the juxtaposition of two permeable moving circles, is one of so much interest and importance that we may as well confirm our result by an independent method: integrating in circles about one of the moving circles; since we may expect the additional energy to vanish for each circular element individually.

Taking the origin at the centre of circle 2, and using polar coordinates, we have,—

$$\frac{u_1u_2+w_1w_2}{V^2a^4}=\frac{-8b^2\sin^2\theta+4br\cos\theta+4b^2+r^2}{r^2(4br\cos\theta+4b^2+r^2)^2}. \quad \text{......(43)}$$

The $rd\theta$ integral of this is,—

$$\frac{-2b\sin\theta}{r^2(4br\cos\theta+4b^2+r^2)}+\frac{\theta}{2r^3}-\frac{1}{r^3}\tan^{-1}\frac{2b-r}{2b+r}\tan\frac{\theta}{2}. \quad \text{......(44)}$$

The value of this when $\theta=0$, and also when $\theta=\pi$, is zero. The additional energy is therefore zero round each circle between $r=a$ and $r=b$. Beyond this, to complete the area over quarter space, we have, when $\theta=\cos^{-1}\frac{-b}{r}$,—

$$\frac{-2b\sqrt{r^2-b^2}}{r^5}+\frac{1}{r^3}\cos^{-1}\frac{b}{r}.$$

For we know that $\frac{\theta}{2}$ lies between $\frac{\pi}{4}$ and $\frac{\pi}{2}$; so $\tan\frac{\theta}{2}$ is $\frac{\sqrt{r+b}}{\sqrt{r-b}}$.

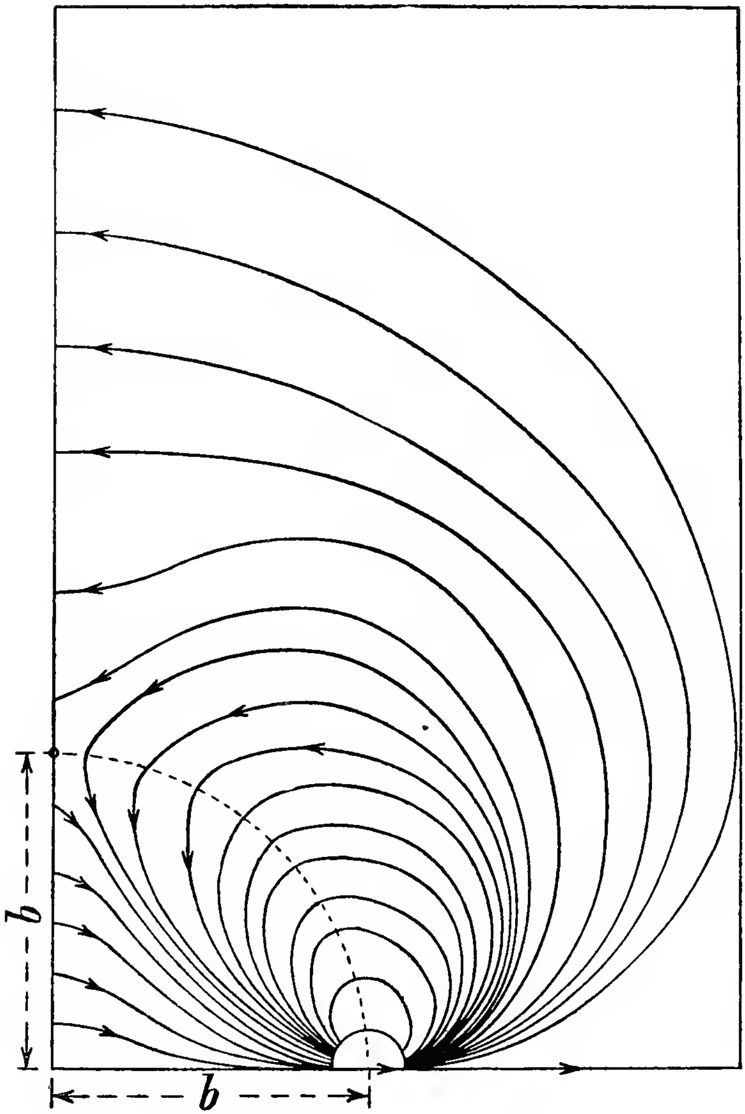

FIG. 9.—CIRCLES IN SEQUENCE DIRECT.

Integrating to r, this yields between b and ∞,—

$$\left[_b^\infty \frac{b\sqrt{r^2-b^2}}{2r^4}-\frac{1}{2r^2}\cos^{-1}\frac{b}{r}\right],$$

and this, without ambiguity, is zero, too.

The $2b-r$ which occurs in the integral to θ, (44), is obtained from $\sqrt{4b^2+r^2-4br}$, and we might, of course, have written this $r-2b$. To have done so, however, would not have resulted in the integral of our function (43), except in the special case when $2b=r$: and in this case the term considered is zero, and on *either* side of this zero value $2b-r$ is required for the integral; therefore the fact that $r-2b$ would satisfy at this one particular point simply indicates that another function's integral crosses that of this at the point considered.

It remains to correct for the fact that each circle actually obstructs the flow from the other. The effect at every point in the fluid of this obstruction is the same as that of a circle (4) at $z=+b$ moving with a velocity $-mV$, where mV is the effective velocity of the fluid at circle 2, due to the motion of circle 1: and a circle at $z=-b$ moving with a velocity $-mV$; and so on. Or $u_4=-mu_2$; $u_3=-mu_1$; $u_6=m^2u_2$; $u_5=m^2u_1$; $u_8=-m^3u_2$; and $u_7=-m^3u_1$.

So for the total energy we have,—

$$\int\frac{u_1^2+u_2^2}{2}, \text{etc.}=\frac{V^2a^2\pi}{2}\times 2: \qquad \int u_1u_2+u_1u_4, \text{ etc.}+u_3u_4, \text{ etc.}=0.$$

$$\int\frac{u_3^2+u_4^2}{2} \quad = \quad ,, \quad \times 2m^2: \quad \int u_1u_5+u_2u_6=\frac{V^2a^2\pi}{2}\times 4m^2.$$

$$\int u_1u_3+u_2u_4 \quad = \quad ,, \quad \times -4m:$$

$$\int u_1u_7+u_2u_8+u_3u_5+u_4u_6=\frac{V^2a^2\pi}{2}\times -8m^3.$$

Or for half space; circles in sequence direct,

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1-2m+3m^2-4m^3\}. \quad \text{.............}(45)$$

Similarly for circles in sequence reversed we obtain,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1+2m+3m^2+4m^3\}. \quad \text{..............}(46)$$

If we take for m the ratio given by assuming the effective velocity is that proper to the centre, or $m=e^2$; we obtain for circles in sequence direct,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1-2e^2+3e^4-4e^6, \text{etc.}\}=\frac{V^2a^2\pi}{2}\cdot\frac{1}{(1+e^2)^2} \quad (47)$$

and $$\frac{dE}{dd}=\frac{V^2a^4\pi}{d^3}\{4-12e^2+24e^4, \text{etc.}\}=\frac{V^2a^4\pi}{d^3}4\frac{1}{(1+e^2)^3}\cdot\ldots\ldots(48)$$

Similarly for circles in sequence reversed,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1+2e^2+3e^4, \text{etc.}\}=\frac{V^2a^2\pi}{2}\cdot\frac{1}{(1-e^2)^2}\ldots\ldots\ldots\ldots(49)$$

and

$$\frac{dE}{dd}=-\frac{V^2a^4\pi}{d^3}\{4+12e^2+24e^4, \text{etc.}\}=-\frac{V^2a^4\pi}{d^3}4\frac{1}{(1-e^2)^3}, \quad (50)$$

since all the correctional circles have necessarily the same directional velocity as that of the original circle with which they coincide.

These are forces which vary inversely as the third power of the distance, modified at close quarters. By reference to fig. 9 we see,—what we might have concluded without it,—that, in sequence direct the lines of flow from the front of the leading circle are drawn round towards the amplification of space represented by the back of the rear circle. Thus the crests of the enclosed lines of flow are pressed over towards the points $x=\pm b$; so that the curvature of the lines of flow on the rear of the front circle and on the front of the rear circle is less than on the outer sides of both. Thus the force is an attraction, since the circles are impermeable and the difference of pressure can act on them.

In the case of sequence reversed this difference in curvature is of course reversed, as shown by the dotted lines in fig. 7. Here, therefore, the force is a repulsion.

If we wish to substitute the mean velocity for mV, we require the discharge through the semicircle at $z=d$, due to the motion of a circle at the origin.

$$\text{This discharge}=Va^2\int_{0;0}^{a}\frac{d^2+a^2-2x^2+2d\sqrt{a^2-x^2}}{(d^2+a^2+2d\sqrt{a^2-x^2})^2}\,dx.$$

Put $x = a \sin\theta$. $\therefore$ $a\cos\theta = \sqrt{a^2 - x^2}$, and $\dfrac{dx}{d\theta} = a\cos\theta$,

giving, $$Va^2 \int_{0 \text{ and } \pi}^{\frac{\pi}{2}} \frac{2a^3\cos^3\theta + 2da^2\cos^2\theta + (d^2 - a^2)a\cos\theta}{(2da\cos\theta + d^2 + a^2)^2} d\theta$$

$$= \frac{Va^3}{2d^3}\left[d\sin\theta - a\theta + \frac{d\sin\theta(d^2+a^2)}{(2da\cos\theta + d^2 + a^2)} - 2a\tan^{-1}\frac{(d-a)\tan\frac{\theta}{2}}{(d+a)} \right]_{0 \text{ and } \pi}^{\frac{\pi}{2}}. \quad \text{......(51)}$$

The function in the bracket is,—when $\theta = 0$, zero; when $\theta = \dfrac{\pi}{2}$, $2d - \dfrac{a\pi}{2} - 2a\tan^{-1}\dfrac{d-a}{d+a}$; when $\theta = \pi$, $-2a\pi$.

Now $\tan^{-1}\dfrac{d-a}{d+a}$ is an angle a little less than 45°; put it, therefore, $= \dfrac{\pi}{4} - \phi$; where ϕ will be a very small angle: and we have $\tan\left(\dfrac{\pi}{4} - \phi\right) = \dfrac{d-a}{d+a}$; giving $\tan\phi = \dfrac{a}{d}$. Thus we can write the mean velocity,—

$$\frac{Va^2}{4d^3}\left\{\pi a + 4d - 4a\left(\frac{\pi}{4} - \tan^{-1}\frac{a}{d}\right)\right\}$$

$$= V\left\{e^2 + e^4 - \frac{e^6}{3} + \frac{e^8}{5}, \text{ etc.}\right\}. \quad \text{......(52)}$$

Substituting this value for the effective velocity in (45), we have,—

$$\frac{E}{2} = \frac{V^2a^2\pi}{2}\left\{1 - 2e^2 + e^4 + \frac{8}{3}e^6 - \frac{57}{5}e^8\right\}, \quad \text{...........(53)}$$

for sequence direct: and substituting in (46), we have,—

$$\frac{E}{2} = \frac{V^2a^2\pi}{2}\left\{1 + 2e^2 + 5e^4 + \frac{28}{3}e^6 + \frac{67}{5}e^8\right\}, \quad \text{.........(54)}$$

for circles in sequence reversed.

The energy, due to a moving circle at the origin, enclosed within $x^2 = a^2 - (z-d)^2$ is,—

$$V^2a^4\int_{d-a}^{d+a}dz\int_0^{\sqrt{a^2-(z-d)^2}}\frac{dx}{(x^2+z^2)^2}$$

$$=V^2a^4\int_{d-a}^{d+a}dz\left[\frac{x}{2z^2(x^2+z^2)}+\frac{1}{2z^3}\tan^{-1}\frac{x}{z}\right]_0^{\sqrt{a^2-(z-d)^2}}$$

$$=V^2a^4\left[\frac{-1}{4z^2}\tan^{-1}\frac{\sqrt{a^2-(z-d)^2}}{z}\right]_{d-a}^{d+a}$$

$$+V^2a^4\int_{d-a}^{d+a}\frac{a^2-d^2-2z^2+3dz}{4z^2(a^2-d^2+2dz)\sqrt{a^2-(z-d)^2}}dz.$$

The first item, taken between the limits mentioned, is evidently zero. After separating the second item into convenient fractions, put $z-d=a\cos\theta$; giving $\frac{dz}{d\theta}=-a\sin\theta$ and $\sqrt{a^2-(z-d)^2}=a\sin\theta$; and we obtain,—

$$V^2a^4\int_\pi^0\left(\frac{-1}{4(a\cos\theta+d)^2}+\frac{d}{4(d^2-a^2)(a\cos\theta+d)}\right.$$
$$\left.+\frac{-a^2}{2(d^2-a^2)(2ad\cos\theta+a^2+d^2)}\right)d\theta$$

$$=V^2a^4\left[\frac{a\sin\theta}{4(d^2-a^2)(a\cos\theta+d)}+\frac{-a^2}{(d^2-a^2)^2}\tan^{-1}\frac{(d-a)\tan\frac{\theta}{2}}{d+a}\right]_\pi^0$$

$$=V^2a^4\frac{a^2}{(d^2-a^2)^2}\cdot\frac{\pi}{2}=\frac{V^2a^2\pi}{2}(e^4+2e^6+3e^8,\text{ etc.}).\qquad(55)$$

Subtracting this from (53), we have for circles in sequence direct,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1-2e^2+0e^4\}\qquad(56)$$

and

$$\frac{dE}{dd}=\frac{V^2a^4\pi}{d^3}\{4-0e^2\}.\qquad(57)$$

And, subtracting from (54), we have for circles in sequence reversed,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1+2e^2+4e^4\}\qquad(58)$$

and

$$\frac{dE}{dd}=-\frac{V^2a^4\pi}{d^3}\{4+16e^2\}.\qquad(59)$$

From what precedes (52), we see that the z discharge of our neglected sink-source, is $2Va\pi e^3$. Its solitary energy will

therefore have no term lower than e^6; while it will compound with the original moving circles to produce additional energy of the order e^5. Thus we have had to discard terms in (56) and (58) above e^4; but these formulæ should now be exact as far as shewn.

It is interesting to compare the altering form of the lines of flow as we change from a solitary moving circle to two permeable moving circles; with the absence of appreciable change between this latter case and the surroundings of a pair of impermeable moving circles.

In the first case we have simply a series of circles; all tangent to the axis of z at the centre of the moving circle: two sets, of course, one at each side of the axis. These may be called the 'solitary' lines, as being those due to a solitary moving circle (fig. 8). The energy is constant throughout the circumference of any circle concentric with the moving circle. The numerical values of the velocity and of the momentum are also constant over the same circles; though their direction of course varies.

When we change to the case of two moving permeable circles in sequence direct, roughly sketched in fig. 9; the lines of flow are profoundly modified. The fluid is at rest at two points; $z=0$, $x=\pm b$: between these points it is moving parallel to z and in the direction of V; above and below it moves parallel to z but in the opposite direction: across the circle $r=b$ its motion is radial to the origin, directed towards it in the first and fourth quadrants and away from it in the other two. Between the two moving circles the lines of flow are conchoidal, the highest touching the singular points $x=\pm b$; where there is not, however, molecular rotation since the four lines of flow meeting at the point, taken in pairs, neutralize each other's action.

The apices of the solitary lines were on the perpendicular to the axis of z; but now the lines of flow have their apices on the circle $r=b$; up to a certain line of flow which has the point $z=0$, $x=b$, as its apex; having zero velocity there. The lines of flow exterior to this do not return to the back of that moving circle from which they started; but, passing

over this critical point, close on the back of the following circle. All the lines of flow are, in a sense, spiral: with reversal of inclination on passing the apex. Those which pass from one circle to another are symmetrical, and have a reflex curve joining the two portions: those which return upon the original circle are egg-shaped.

Where these cross the circle $x^2+z^2=b^2$, they are cotangential with the solitary lines of flow of both moving circles. At the apex their curvature is greater than that of the solitary lines; at the moving circle their curvature is less; but the reduction in the curvature of the solitary lines here, *i.e.* at $z=b$, becomes much more marked as we move towards the origin than is the case when we move outwards; for the constraining lines of flow cut the solitary lines of flow we are considering at a greater inclination on the inner side than on the outer; besides being a little more powerful on the inner side. Hence, on any circle of finite magnitude, the curvature of the lines of flow where these lines leave the front of the leading circle or close on the back of the following circle, is greater than that of the corresponding line which closes on the other side: so that the circle,—if impermeable,—is pressed towards the origin.

Now these lines of flow are the 'permeable' lines of flow, constructed by simply compounding the two velocities due to the independent motions of the circles: or indicate the resultant of u_1+u_2 and w_1+w_2. To correct for impermeability we compound $(u_1+u_2+\text{etc. } u_n)$ with $(w_1+w_2+\text{etc. } w_n)$; but, as we have seen when constructing formula (45), u_3, u_5, etc. are obtainable by multiplying u_1 by exactly the same factors which convert u_2 into u_4, u_6, etc., w_1 into w_3, etc., and w_2 into w_4, etc. So what we have to compound is $\phi(d)\{u_1+u_2\}$ and $\phi(d)\{w_1+w_2\}$: $\phi(d)$ being, as a matter of fact in this particular case $(1-m+m^2-m^3, \text{etc.})$: but whatever $\phi(d)$ is, the direction of the resultant velocity cannot be altered except by an integral multiple of π: the direction of the line of flow, considered as a curve, cannot therefore be altered; though the speed in that line will probably be changed, and may even, considered as a velocity, be reversed in direction.

There is therefore no difference in form between the per-

meable and impermeable lines of flow for two circles, or spheres, in sequence; except that due to the neglected sink-source effect.

In the case of two circles in sequence reversed, the change in sign of $\frac{dE}{dd}$ compared with the value of this function for circles in sequence direct,—everything else being symmetrical,—of course shews that the attraction in the case of sequence direct has become a repulsion on the reversal of one circle's velocity. This can, however, also be seen directly from the dotted lines of flow in fig. 7. We see that moving from the centre of the circle at $z=b$ towards the origin we are moving towards the pointed end of the oval and from the extremity of the oval's minor axis where the curvature is a minimum: it therefore follows that pressure on the inner side of the circles is greater than on the outer.

Indeed these diagrams, roughly drawn as they are, still shew how, when the higher powers of e are considered, the repulsion in one case is greater than the attraction in the other: for in fig. 7 our circular 'boundary' is between the points of grand maximum and grand minimum curvatures in the oval, while in fig. 9 it lies between the mid-maximum and the grand minimum.

Let us now consider a pair of circles in parallel. Circle 1, at $x=-b$; circle 2, at $x=+b$; both moving with velocity V parallel to z. We have,—

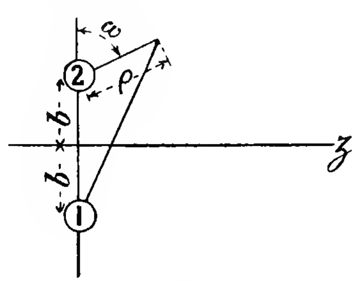

FIG. 10.

$$u_2 = Va^2 \frac{2z(x-b)}{(z^2+(x-b)^2)^2}; \qquad u_1 = Va^2 \frac{2z(x+b)}{(z^2+(x+b)^2)^2}.$$

$$w_2 = Va^2 \frac{z^2-(x-b)^2}{(z^2+(x-b)^2)^2}; \qquad w_1 = Va^2 \frac{z^2-(x+b)^2}{(z^2+(x+b)^2)^2};$$

or, putting $x=b+r\cos\omega$; and $z=r\sin\omega$, for $\dfrac{u_1u_2+w_1w_2}{V^2a^4}$,

we get
$$\frac{-8b^2\sin^2\omega+4br\cos\omega+4b^2+r^2}{r^2(4br\cos\omega+4b^2+r^2)^2};$$

or identically the same expression (43) which we had for our additional energy function in the case of two 'following' circles; and the limits of integration for ω here will be the same as those for θ there: while the limits for r are unchanged. The additional energy is therefore—for permeable circles,—zero here as it was in the other case.

To modify for impermeability; if $-mV$ stand for the effective velocity against the obstructing circle at $x=+b$, due to the motion of that at $x=-b$; since all the correctional circles have the same directioned velocity as that of the original circle with which they coincide, we have for circles in parallel direct,

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1+2m+3m^2+4m^3, \text{etc.}\}, \quad\text{............(60)}$$

which is identical with the expression for circles in sequence reversed: and for circles in parallel reversed we obtain the expression for circles in sequence direct.

The ratio for 'central' velocity, is here identical with that in the case of circles in sequence, *i.e.* is e^2: and we obtain for circles in parallel direct (49) and (50); identical with those for sequence reversed. Similarly for circles in parallel reversed we have (47) and (48), as in the case of sequence direct.

From fig. 11 we see, that in the case of parallel direct, many lines of flow starting from the near side of the moving circle, are deflected over to the far side; so that the curvature of the lines of flow on the far side is considerably increased where they spring from the circle; truly, the curvature of such as remain below $r=b$ on the near side is also increased,—sharing in the effect of the positive constraint; but, for some little distance below $r=b$, less increased than is the curvature on the far side, because the deflecting

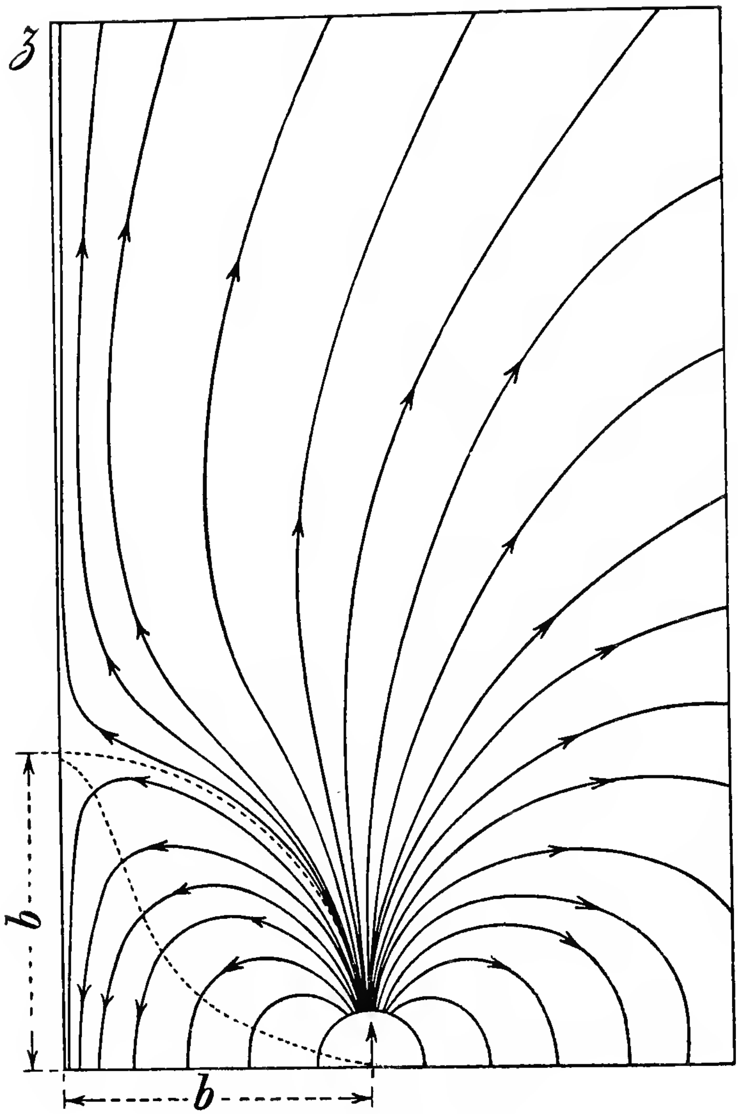

Fig. 11.—Circles in Parallel Direct.

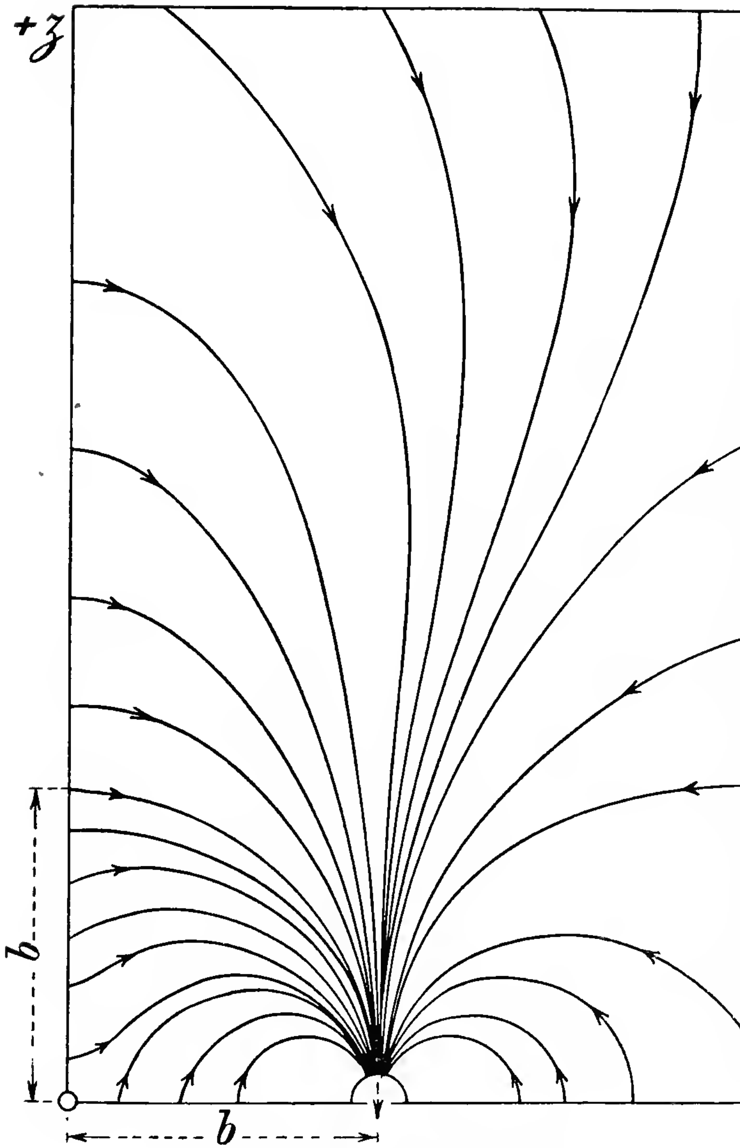

FIG. 12.—CIRCLES IN PARALLEL REVERSED.

lines of flow are here more nearly coincident in direction with the solitary lines; and below this point, where the curvature of the deformed lines on the inner side is greater than on the far side, the components parallel to x of the velocities in all these lines are considerably smaller than those in the corresponding lines on the outer side; and all the near side lines of flow above $r=b$, have their curvature diminished; hence the force is an attraction between the circles.

From fig. 12 we see that, in the case of parallel reversed, many of the lines of flow which—when solitary—came round on the outer side, now come round from the inner; or from the other circle's outer half; and in other respects conditions holding in the case of parallel direct are here reversed: hence the force is a repulsion.

In order to obtain the mean velocity to substitute for $-mV$, we must integrate the value of wdx along the semicircle of $z^2=a^2-(x-d)^2$, between $x=d-a$ and $x=d+a$: for in this case it is the front and back of the circle that are symmetrical; and the two sides that differ as to the lines of flow which permeate them. Or, expressing wdx in terms of ω and a, we can integrate from π to 0.

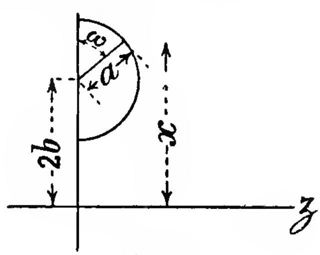

FIG. 13.

Putting $x=d+a\cos\omega$, $\therefore\ \dfrac{dx}{d\omega}=-a\sin\omega$, we have discharge

$$=Va^3\int_\pi^0 \frac{2a^2\cos^2\omega+2da\cos\omega+d^2-a^2}{(2da\cos\omega+d^2+a^2)^2}\sin\omega\,d\omega$$

$$=\frac{Va^3}{2d^2}\Big[-\cos\omega+\frac{(d^2-a^2)^2}{2da(2da\cos\omega+d^2+a^2)}$$

$$+\frac{a}{d}\log(2da\cos\omega+d^2+a^2)\Big]_\pi^0,$$

giving mean velocity,—

$$-V\left(e^2-e^4-\frac{e^6}{3}-\frac{e^8}{5}, \text{ etc.}\right) \quad\text{.................}(61)$$

for circles in parallel.

If we substitute the expression in the bracket for m in (60), we have for circles in parallel direct,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\left\{1+2e^2+e^4-\frac{8}{3}e^6-\frac{57}{5}e^8\right\}, \quad\text{...........}(62)$$

and, similarly, from (45) can be obtained for circles in parallel reversed,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\left\{1-2e^2+5e^4-\frac{28}{3}e^6+\frac{67}{5}e^8\right\}. \quad\text{..........}(63)$$

The 'enclosed energy' is the same as in the case of sequence; for the solitary isoenergic lines are circles. Subtracting, then, (55) from the above, we obtain for parallel direct,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1+2e^2+0e^4\}, \quad\text{.................}(64)$$

$$\frac{dE}{dd}=-\frac{V^2a^4\pi}{d^3}\{4+0e^2\}, \quad\text{....................}(65)$$

and for parallel reversed,—

$$\frac{E}{2}=\frac{V^2a^2\pi}{2}\{1-2e^2+4e^4\}, \quad\text{.................}(66)$$

$$\frac{dE}{dd}=\frac{V^2a^4\pi}{d^3}\{4-16e^2\}. \quad\text{......................}(67)$$

Thus we see that the attraction between circles in parallel direct equals the attraction between circles in sequence direct, so far as we have gone: but the repulsion between circles in parallel reversed, when at close quarters, is less than the repulsion between circles in sequence reversed at the same distance.

If we evaluate what precedes (61), between $\frac{\pi}{2}$ and 0; we find that, after allowing for the mean velocity parallel to z, there remains a z-discharge in the first and second quadrants, each of $Va(e^3-2e^5)$; and in the third and fourth quadrants of $-Va(e^3-2e^5)$. Our neglected x-discharge gives similar

quadrant sink-sources: and, whether we choose to consider these separately or compound them together, the effect of their suppression is four lobed lines of flow as before; but in this case the axes of the lobes are parallel to the coordinate axes, instead of being at 45° with these directions. The orientation of a system,—though this may affect numerical coefficients,—has no effect upon the order in e: so these sink-sources compound as before to no lower term than e^5.

In connection with the lines of flow for two moving circles or spheres; and for two sinks; there are two loci worth examination: one bearing a certain relation to the amount of force exerted; and one which is of much assistance when we seek to determine the sign of the force. The *orthogonal* curve is the locus of rectangular intersections between solitary lines of flow; and marks the zero of the 'additional energy': on crossing it we find the additional energy change sign: on one side of it the angle between the velocities to be compounded is acute; on the other obtuse.

In the case of two circles in sequence this curve is a pair of circles, radii $b\sqrt{2}$, and centres at $x=\pm b$: shewn by the broken line in fig. 8.

In the case of two spheres in sequence, fig. 1, the trace of the orthogonal surface on xz is,

$$x^4+x^2(5z^2-13b^2)+4(z^2-b^2)^2=0.$$

This may be written,—

$$\{x^2-x\sqrt{9b^2-z^2}+2(z^2-b^2)\}\{x^2+x\sqrt{9b^2-z^2}+2(z^2-b^2)\}=0,$$

which shews that the surface is described by the revolution about z of either of these two curves. They are ovals, not unlike inverted solitary lines of flow, but longer in comparison with their width.

When two circles are in parallel, as we have seen, the additional energy expression is identical with that for circles in sequence,—if we exchange x and z. So here the orthogonal curve is a pair of circles radii $b\sqrt{2}$, and centres at $z=\pm b, x=0$.

For two spheres in parallel we have for the trace on xz, $4z^4+z^2(5x^2-13b^2)+(x^2-b^2)^2=0$; or if we like,—

$$\{2z^2-z\sqrt{9b^2-x^2}+(x^2-b^2)\}\{2z^2+z\sqrt{9b^2-x^2}+(x^2-b^2)\}=0,$$

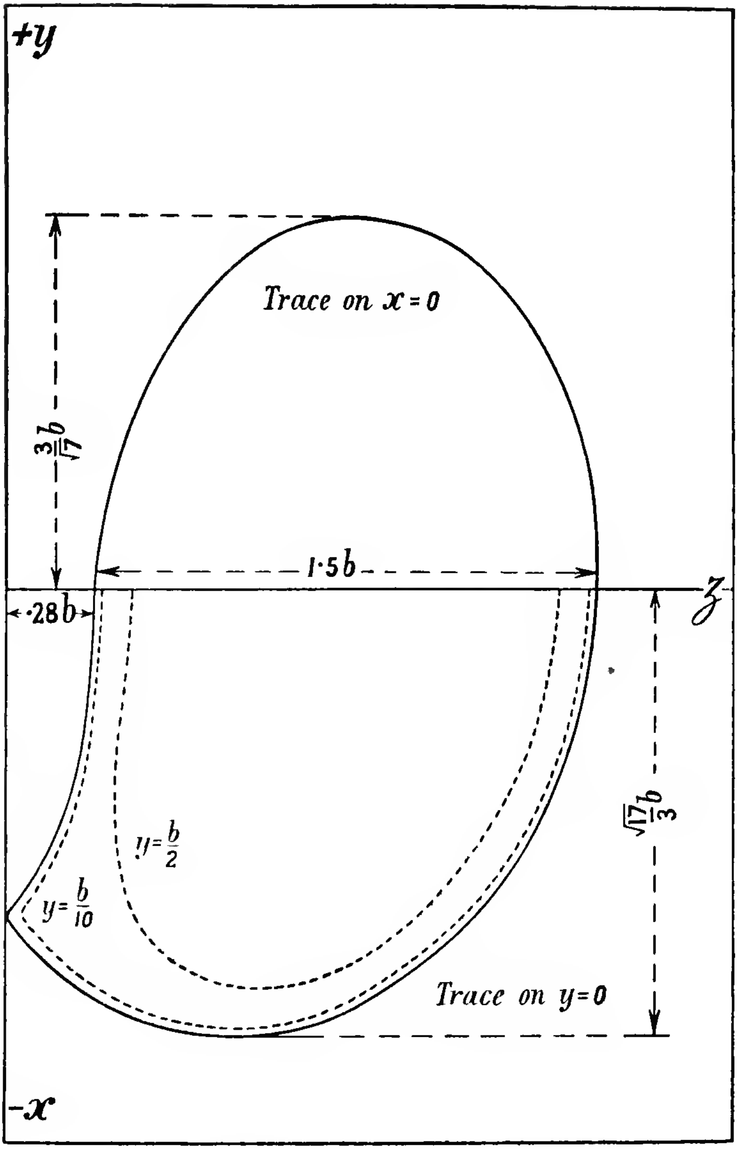

FIG. 14.—ORTHOGONAL SURFACE: SPHERES IN PARALLEL.

which gives a curve whose ordinates are just half those of the curve for sequence: the 'ordinate' in the case of sequence being called x; and in the case of parallel z. This dimension of the orthogonal curve perpendicular to the line joining the moving particles, is related to the force exerted upon the particles by the fluid. Here, of course, we have not a surface of revolution; but a curiously shaped surface having two horns. The above trace on xz is shewn in the lower half of fig. 14, together with dotted traces on planes $y=\frac{b}{10}$ and $y=\frac{b}{2}$, to exhibit the way in which the horns retract as we leave the plane of xz. The trace on the plane yz, is,—

$$2z^2-z\sqrt{17b^2-y^2}+y^2+b^2=0\,;$$

a curve resembling an ellipse; but really not even possessing symmetry about more than one axis: it is shewn in the upper part of the same figure. A pair of spherical sinks has the sphere $x^2+y^2+z^2=b^2$ for its orthogonal surface. A *two-dimensional* sink in an incompressible fluid is a physical impossibility; as its energy would be infinite.

A spherical sink and a moving sphere,—a combination we had to consider when correcting the action between two sinks for impermeability,—have between them for their orthogonal locus a surface of revolution whose trace on xz is a curve in z^3, which is, of course, not symmetrical about x. It can be written $x^2(z+2b)+(z-b)(z+b)^2=0$; shewing that it is a looped curve passing through $z=+b$, where the sink is; crossing at $z=-b$, where the moving sphere is; and then running off to infinity in two branches asymptotic to $z=-2b$. A circular sink and moving circle in two dimensions would have a similar figure for their orthogonal locus: but in this case the asymptote would be $z=-3b$; and the semiaxis minor of the loop would be $\frac{b}{\sqrt{3}}$ instead of $\frac{b}{\sqrt{2}}$.

If in order to integrate easily the 'additional energy,' we strike out those terms which cancel to right and left of the origin,—*i.e.* alter to a mean position our zero or datum; the

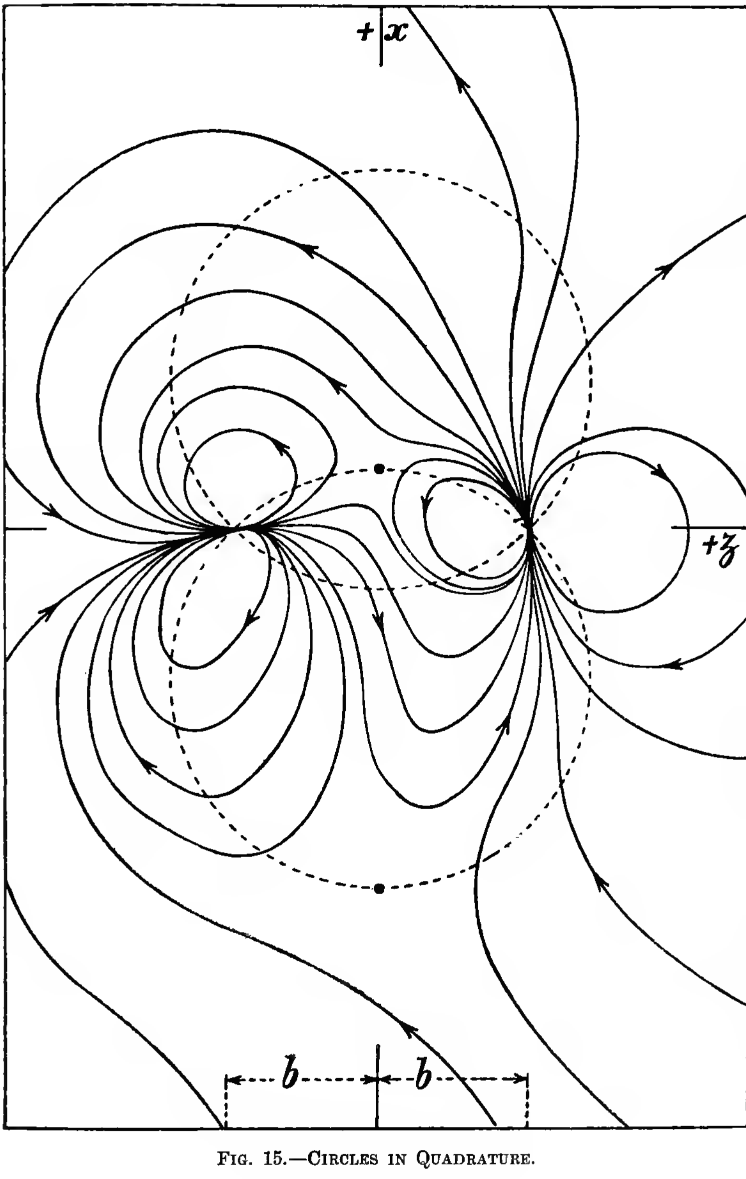

FIG. 15.—CIRCLES IN QUADRATURE.

zero surface for additional energy in the three-dimensional problem has for its trace on xz,

$$2x^4+x^2(z^2+b^2)-(z^2-b^2)^2=0\,;$$

which may be written

$$\{2z^2-x^2-2b^2+x\sqrt{9x^2+8b^2}\}\{2z^2-x^2-2b^2-x\sqrt{9x^2+8b^2}\}=0\,;$$

a pair of conchoidal curves, by the revolution of either of which about z, we get the zero additional energy surface. These conchoids have their vertices at $x=\pm\dfrac{b}{\sqrt{2}}$; cross at $z=\pm b$; and, after crossing, have so slight a reflex curvature that they might almost be mistaken for hyperbolas. The cone $z=\sqrt{2}x$ is asymptotic.

Similarly in two dimensions we would have

$$3x^4+2x^2(b^2+z^2)-(z^2-b^2)^2=0\,;$$

or $$\{z^2-x^2-b^2+2x\sqrt{x^2+b^2}\}\{z^2-x^2-b^2-2x\sqrt{x^2+b^2}\}=0\,;$$

two similar curves; vertices at $x=\pm\dfrac{b}{\sqrt{3}}$; crossing at $z=\pm b$; and having $z=\pm x\sqrt{3}$ as asymptotes.

When we have two moving circles in what may be termed quadrature; the first at $z=-b$ moving towards the second at $z=+b$; the second moving along $+x$: the orthogonal curve is the circle $x^2+z^2=b^2$, and the axis of z.

What may be called the *determinant* curve, is a still more important locus when we wish to trace the form of the lines of flow, and especially that variation in their curvature which determines the sign of the resultant force. This curve is the locus of tangent points between the two solitary systems; and therefore the change in the direction of the velocity there, consequent upon the introduction of the second system to the first, is either 0 or π: that is to say, the three systems,—two solitary and one compound,—all have a common tangent where they cross this curve.

The least consideration of the direction of motion in the solitary lines on each side of this curve, enables us to see whether the compound line of flow at this point touches the

corresponding solitary line inside it or outside: *i.e.* whether its curvature is greater or less than before.

In the case of two circles in sequence, this locus is the circle $x^2+z^2=b^2$; and the axis of z; and the line or circle at infinity $x^2+z^2=\infty$: the latter two being 'coincident' branches. When the sequence is 'direct,' fig. 9, at the point $z=b$,—the centre of the moving circle,—the compound lines would touch the dominant solitary (or that solitary which has the greater curvature) on the outside of the solitary line; elsewhere on the determinant circle they touch on the inside, that forming the pointed end of the oval: on the 'coincident' branches they coincide.

When the sequence is 'reversed,' fig. 7, dotted lines, they touch on the outside at both points: the pointed end of the oval being directed towards the origin.

In the case of two spheres in sequence we have the ellipse $\frac{z^2}{b^2}+\frac{x^2}{2b^2}=1$, as the trace of their determinant surface, which surface is, of course, the oblate spheroid described by the revolution of this ellipse about the axis of z: and the axis of z and the sphere or plane at ∞, are again 'coincident' branches. Thus the figures which are drawn for circles in sequence serve,—so far as general descriptive purposes are concerned,—for the three-dimensional problem as well, if we imagine the determinant circle drawn out into an ellipse whose semiaxis major is $b\sqrt{2}$: the compound lines of flow being distorted to suit. For two circles in parallel, the determinant curve is the axis of x as a general branch; and the circle $x^2+z^2=b^2$, together with the circle at infinity, as particular 'coincident' branches.

On the general branch in parallel direct, fig. 11, the compound lines of flow touch the dominant solitary lines on the outside; in parallel reversed, fig. 12, on the inside.

Two spheres in parallel have the plane of xy and the circle at ∞ in xz and the ellipse $\frac{x^2}{b^2}+\frac{2z^2}{b^2}=1$, as determinant loci: the last two not representing surfaces, but simply lines. Thus the particular loci where the fluid is at rest, are, for parallel

reversed, the axis of y; and for parallel direct, the two *points* $z=\pm\frac{b}{\sqrt{2}}$, $x=0$, $y=0$: not a point and circle respectively, as they were in the case of spheres in sequence.

A pair of sinks or sources, or a sink and a source, one at $z=+b$, and one at $z=-b$; would have for their determinant curves in two dimensions, the circle $r=\infty$ as a general branch; and the axis of z as a particular coincident branch. In three dimensions we have the sphere at ∞ and the axis of z.

'Determinant' loci are not, however, of the same use in the case of sinks and sources as they are in the determination of force direction for moving particles. In the case of a moving particle, a certain discharge must leave a certain area element of space which is at a given instant occupied by a particular element of a moving particle's surface, if that particle is to move in a certain direction with a certain speed: that is equivalent to asserting that the end of the line of flow is fixed on the particle. Variation of curvature is therefore of paramount importance as determining the variation of fluid pressure there.

In the case of a sink, however, there is no absolute necessity for particular portions of the discharge to enter the sink at any particular portions of its circumference: yielding to any constraint, the ends of the lines of flow are free to shift on the surface. It is the density of the crowding of these lines of flow; or the amount of momentum entering at any side, which determines the force.

In the case of a circle at $z=-b$ moving towards or from a sink at $z=+b$, in two dimensions, we have the general locus $x^2+(z-b)^2=4b^2$; and, as a particular 'coincident' locus, the axis of z. In three dimensions these become the sphere $x^2+(z-2b)^2=9b^2$; and the axis of z. One might not necessarily have expected the circular trace, here; though of course it was obvious the centre of curvature must be further from the origin than is the sink.

For two circles in 'quadrature,' the determinant curve is the pair of circles $(x\pm b)^2+z^2=2b^2$, as a general branch, and the circle at ∞ as a 'coincident' branch. The fluid is at rest at

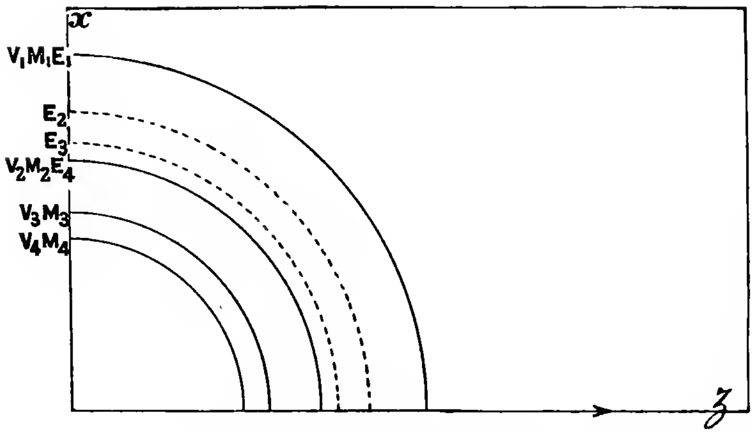

FIG. 16.—SINGLE CIRCLE CONTOURS.

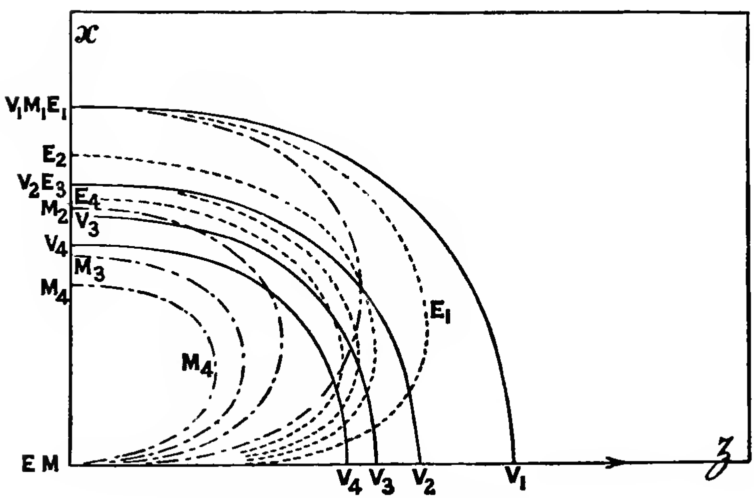

FIG. 17.—SINGLE SPHERE CONTOURS.

two unsymmetrically situated spots, $z=0$, $x=b(-1\pm\sqrt{2})$. If we call that solitary line of flow which has the greater curvature, the 'dominant' solitary,—whether it has the greater speed or not,—we see that over the whole circle $(x+b)^2+z^2=2b^2$, the lower circle in fig. 15, the compound line of flow has a greater curvature than the dominant solitary,—touches this on the inner side; and over the whole of the circle $(x-b)^2+z^2=2b^2$, a smaller curvature; touching the dominant solitary on its outside.

Fig. 16 shews a few contours for a solitary moving circle. The full lines are speed contours, which here coincide with equimomental lines; and the dotted are isoenergic lines. These are, of course, all circles having the centre of the moving circle for centre.

Similar contours for the three-dimensional case are shewn in fig. 17. The speed contour is the trace on xz,—the plane of the diagram,—of the corresponding surface of revolution; but the shattered and the dotted lines are not traces of the equimomental and isoenergic surfaces; such would coincide with certain speed contour traces, and their representation would be of little use. The contours shewn for momentum and energy are drawn on the supposition that all the mass of fluid, corresponding in three dimensions to any given area on xz, is collected into that area on that plane. Thus, when considering the comparative effect of the lines of flow from some other sphere crossing the equatorial region and the polar regions, we see what relative importance attaches to each portion of our plane diagram. The maximum momentum is at about co-latitude 54° 44′; and for energy, about 41° 46′.

VI

SPHERES IN PARALLEL

We have yet to consider the case of two spheres in parallel,—say sphere 1, at $x=-b$, and sphere 2, at $x=+b$: each moving with velocity V parallel to the axis of z. Writing v for the

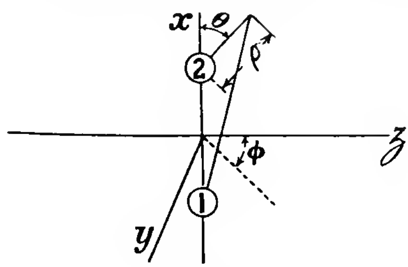

Fig. 18.

rectangular component velocity along the axis of y, and using polar coordinates as shewn in fig. 18, with sphere 2 as origin; we have,—

$$u_2 = Va^3 \frac{3\cos\theta\sin\theta\cos\phi}{2\rho^3};$$

$$v_2 = Va^3 \frac{3\sin^2\theta\sin\phi\cos\phi}{2\rho^3};$$

$$w_2 = Va^3 \frac{3\cos^2\phi\sin^2\theta - 1}{2\rho^3};$$

$$u_1 = Va^3 \frac{3(d\rho\sin\theta + \rho^2\sin\theta\cos\theta}{2\{2\rho d\cos\theta + d^2 + \rho^2\}^{\frac{5}{2}}}\cos\phi;$$

$$v_1 = Va^3 \frac{3\rho^2\sin^2\theta\sin\phi}{2\{\ ,,\ \}^{\frac{5}{2}}}\cos\phi;$$

$$w_1 = Va^3 \frac{3\rho^2\sin^2\theta\cos^2\phi - 2d\rho\cos\theta - d^2 - \rho^2}{2\{\ ,,\ \}^{\frac{5}{2}}}.$$

Or $\quad 4\dfrac{u_1u_2+v_1v_2+w_1w_2}{V^2a^6}$

$$=\frac{3\cos^2\phi(-d\rho\cos^3\theta+\cos^2\theta(d^2-\rho^2)+d\rho\cos\theta+\rho^2-d^2)}{\rho^3\{2\rho d\cos\theta+d^2+\rho^2\}^{\frac{5}{2}}}$$

$$+\frac{1}{\rho^3\{\ ,,\ \}^{\frac{3}{2}}}=A,\text{ say,}$$

$$=\frac{\cos^2\phi}{8d^2\rho^5}\left\{-3\{\ ,,\ \}^{\frac{1}{2}}+\frac{15d^2+3\rho^2}{\{\ ,,\ \}^{\frac{1}{2}}}+\frac{-21d^4-6d^2\rho^2+3\rho^4}{\{\ ,,\ \}^{\frac{3}{2}}}\right.$$

$$\left.+\frac{9d^6-21d^4\rho^2+15d^2\rho^4-3\rho^6}{\{\ ,,\ \}^{\frac{5}{2}}}\right\}+\frac{8d^2\rho^2}{8d^2\rho^5\{\ ,,\ \}^{\frac{3}{2}}}.\ ..(68)$$

Linear integral of above to $\rho\sin\theta d\phi$, between $\phi=0$, and $\phi=\frac{\pi}{2}$; since $\int_0^{\frac{\pi}{2}}\cos^2\phi d\phi=\frac{\pi}{4}$, is,—

$$\frac{\pi\rho\sin\theta}{32d^2\rho^5}\left\{-3\{\ ,,\ \}^{\frac{1}{2}}+\frac{15d^2+3\rho^2}{\{\ ,,\ \}^{\frac{1}{2}}}+\frac{-21d^4+10d^2\rho^2+3\rho^4}{\{\ ,,\ \}^{\frac{3}{2}}}\right.$$

$$\left.+\frac{9d^6-21d^4\rho^2+15d^2\rho^4-3\rho^6}{\{\ ,,\ \}^{\frac{5}{2}}}\right\}.\ \ldots\ldots\ldots\ldots(69)$$

The integral of this to $\rho d\theta$, is,—

$$\frac{\pi}{32d^3\rho^4}\left\{\{\ ,,\ \}^{\frac{3}{2}}+(-15d^2-3\rho^2)\{\ ,,\ \}^{\frac{1}{2}}+\frac{-21d^4+10d^2\rho^2+3\rho^4}{\{\ ,,\ \}^{\frac{1}{2}}}\right.$$

$$\left.+\frac{3d^6-7d^4\rho^2+5d^2\rho^4-\rho^6}{\{2d\rho\cos\theta+d^2+\rho^2\}^{\frac{3}{2}}}\right\}.\ \ldots\ldots\ldots(70)$$

The value of this when $\theta=0$ is $\dfrac{-\pi}{\rho^4}$; and when $\theta=\pi$ is also $\dfrac{-\pi}{\rho^4}$. Therefore $\int_0^{\pi}\rho d\theta\int_0^{\frac{\pi}{2}}A\rho\sin\theta d\phi=0$, always.

The value of (70) when $\theta=\cos^{-1}\dfrac{-d}{2\rho}$, or up to the plane yz, is $\pi\dfrac{3d^3-28\rho^2d}{32\rho^7}$; therefore the integral between $\theta=0$ and $\theta=\cos^{-1}\dfrac{-d}{2\rho}$ is

$$\frac{\pi}{\rho^4}-\frac{7\pi d}{8\rho^5}+\frac{3\pi d^3}{32\rho^7},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(71)$$

the integral of which to ρ, from $\dfrac{d}{2}$ to ∞, is $\dfrac{\pi}{6d^3}$.

The additional energy, therefore, over an octant of space is $\frac{V^2a^6\pi}{24d^3}$; since, as we have seen, the integral between $\rho=a$ and $\rho=\frac{d}{2}$, is zero over every spherical surface.

This gives the additional energy, permeable spheres in parallel direct, over half space,—

$$\frac{V^2a^6\pi}{6d^3}. \qquad\qquad (72)$$

or just half what it was for spheres in sequence.

This is, of course, due to the fact that the solitary lines of flow are elongated in the direction of x, which in itself is due to the greater freedom obtained by shift in that direction compared with what freedom is obtained by shift in the direction of the spheres' motion. Thus the velocity at a given distance is less when this distance is measured along x, than when it is measured along z: in fact, just half the amount.

To allow for impermeability; as before, let $-mV$ be the effective velocity parallel to z at site of sphere 2 due to the motion of sphere 1. The velocity of correctional sphere 4 will then be $+mV$; or $u_4=mu_2$; $u_3=mu_1$; $u_6=m^2u_2$; $u_5=m^2u_1$; $u_8=m^3u_2$; $u_7=m^3u_1$: and we have,—

$$\int\frac{u^2+u_2^2}{2}=\frac{V^2a^3\pi}{3}\times 2; \qquad \int u_1u_2=\frac{V^2a^6\pi}{3d^3}.$$

$$\int\frac{u_3^2+u_4^2}{2}= \quad ,, \quad \times 2m^2; \qquad \int u_1u_4+u_2u_3= \quad ,, \quad \times 2m.$$

$$\int u_1u_3+u_2u_4= \quad ,, \quad \times 4m; \qquad \int u_1u_6+u_2u_5+u_3u_4= \quad ,, \quad \times 3m^2.$$

$$\int u_1u_5+u_2u_6= \quad ,, \quad \times 4m^2;$$

$$\int u_1u_7+u_2u_8+u_3u_5+u_4u_6=\frac{V^2a^3\pi}{3}\times 8m^3,$$

giving

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}(1+2m+3m^2+4m^3)+\frac{V^2a^3\pi}{3}\cdot\frac{e^3}{2}(1+2m+3m^2) \quad (73)$$

for spheres in parallel direct.

If we reverse the velocity of one of the spheres, we similarly obtain,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}(1-2m+3m^2-4m^3)-\frac{V^2a^3\pi}{3}\cdot\frac{e^3}{2}(1-2m+3m^2). \quad (74)$$

If we take the 'central' velocity for m, or make $m=\frac{e^3}{2}$, we obtain for spheres in parallel direct

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left(1+\frac{3}{2}e^3+\frac{5}{4}e^6+\frac{7}{8}e^9\right)=\frac{V^2a^3\pi}{3}\cdot\frac{2+e^3}{(2-e^3)^2}, \quad \dots\dots\dots(75)$$

$$\frac{dE}{dd}=-\frac{V^2a^6\pi}{d^4}\left(3+5e^3+\frac{21}{4}e^6+\frac{9}{2}e^9\right)=-\frac{V^2a^6\pi}{d^4}\cdot 4\cdot\frac{6+e^3}{(2-e^3)^3}, \quad (76)$$

and similarly for spheres in parallel reversed,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left(1-\frac{3}{2}e^3+\frac{5}{4}e^6-\frac{7}{8}e^9\right)=\frac{V^2a^3\pi}{3}\cdot\frac{2-e^3}{(2+e^3)^2}, \quad \dots\dots(77)$$

$$\frac{dE}{dd}=\frac{V^2a^6\pi}{d^4}\left(3-5e^3+\frac{21}{4}e^6-\frac{9}{2}e^9\right)=\frac{V^2a^6\pi}{d^4}\cdot 4\cdot\frac{6-e^3}{(2+e^3)^3}. \quad (78)$$

In order to substitute the mean velocity as a better value for mV, we require the integral discharge parallel to z, over the hemisphere of $z^2=a^2-x^2-y^2$, due to the motion of a sphere at $x=-d$; that is,—

$$-Va^3\int_{-a}^{+a}dx\int_0^{\sqrt{a^2-x^2}}dy\frac{3x^2+2xd+d^2+3y^2-2a^2}{\{2xd+d^2+a^2\}^{\frac{5}{2}}}.$$

Putting $x=a\cos\theta$; $y=a\sin\theta\sin\phi$, we have $\frac{dx}{d\theta}=-a\sin\theta$; $\frac{dy}{d\phi}=a\sin\theta\cos\phi=z$: and when $y=0$, $\phi=0$; when $y=\sqrt{a^2-x^2}$, $\phi=\frac{\pi}{2}$; and when $x=-a$, $\theta=\pi$; when $x=+a$, $\theta=0$.

Therefore the discharge over a hemisphere is,—

$$-Va^5\int_\pi^0 d\theta\int_0^{\frac{\pi}{2}}d\phi\,\frac{-2ad\cos\theta-d^2-a^2+3a^2\sin^2\theta\cos^2\phi}{(2ad\cos\theta+d^2+a^2)^{\frac{5}{2}}}\sin^2\theta\cos\phi$$

$$=-Va^3\int_\pi^0 d\theta\left(\frac{\{\;,,\;\}^{\frac{3}{2}}}{8d^4}+\frac{-(d^2+2a^2)\{\;,,\;\}^{\frac{1}{2}}}{4d^4}+\frac{d^4+3a^4}{4d^4\{\;,,\;\}^{\frac{1}{2}}}\right.$$

$$\left.+\frac{-d^6+3d^2a^4-2a^6}{4d^4\{\;,,\;\}^{\frac{3}{2}}}+\frac{d^8-4d^6a^2+6d^4a^4-4d^2a^6+a^8}{8d^4\{2ad\cos\theta+d^2+a^2\}^{\frac{5}{2}}}\right).$$

Now dividing by πa^2 to get the mean velocity; and dividing out the d^4, and putting $\frac{a}{d}=e$, as before; and writing,—for the sake of brevity, $\beta=\frac{2e}{1+e^2}$, we get for the mean velocity,—

$$-\frac{V(1+e^2)^{\frac{1}{2}}}{4\pi}e\int_\pi^0 d\theta\Big(\frac{(1+e^2)\{1+\beta\cos\theta\}^{\frac{3}{2}}}{2}+(-1-2e^2)\{\ ,,\ \}^{\frac{1}{2}}$$

$$+\frac{1+3e^4}{(1+e^2)\{\ ,,\ \}^{\frac{1}{2}}}+\frac{-1+3e^4-2e^6}{(1+e^2)^2\{\ ,,\ \}^{\frac{3}{2}}}+\frac{(1-e^2)^4}{2(1+e^2)^3\{\ ,,\ \}^{\frac{5}{2}}}\Big).\ \ldots(79)$$

Now,

$$(1+\beta\cos\theta)^{\frac{3}{2}}=1+\frac{3}{2}\beta\cos\theta+\frac{3}{2^3}\beta^2\cos^2\theta-\frac{1}{2^4}\beta^3\cos^3\theta$$

$$+\frac{3}{2^7}\beta^4\cos^4\theta-\frac{3}{2^8}\beta^5\cos^5\theta+\frac{7}{2^{10}}\beta^6\cos^6\theta,\ \text{etc.}$$

But $\int_\pi^0 \cos^{2n+1}\theta d\theta=0$ always: and $\int_\pi^0\cos^2\theta d\theta=-\frac{\pi}{2}$; of $\cos^4$ is $-\frac{3\pi}{8}$; of $\cos^6$, is $-\frac{5\pi}{16}$; and $\int_\pi^0 1\,.\,d\theta=-\pi$. Therefore the first item of (79) gives,—

$$\frac{V(1+e^2)^{\frac{1}{2}}}{4}\cdot e\cdot\left\{\frac{1}{2}+\frac{e^2}{2}+\frac{3e^2}{2^3(1+e^2)}+\frac{9e^4}{2^7(1+e^2)^3}+\frac{35\,.\,e^6}{2^9(1+e^2)^5},\ \text{etc.}\right\}.$$

Treating all the other items in a similar manner, and adding together the highly convergent series which result, we have,—

$$\text{Mean velocity}=-V\Big(\frac{e^3}{2}-\frac{9e^5}{16}-\frac{25e^7}{128},\ \text{etc.}\Big)\ \ldots\ldots\ldots(80)$$

for spheres in parallel; correct to the seventh power of $\frac{a}{d}$.

Using this value for $-mV$ in (73), we obtain for spheres in parallel direct,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left\{1+\frac{3}{2}e^3-\frac{9}{8}e^5+\frac{5}{4}e^6-\frac{25}{64}e^7-\frac{9}{4}e^8\right\},\ldots\ldots\ldots(81)$$

and for spheres in parallel reversed,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left\{1-\frac{3}{2}e^3+\frac{9}{8}e^5+\frac{5}{4}e^6+\frac{25}{64}e^7-\frac{9}{4}e^8\right\}.\ \ldots\ldots(82)$$

It remains to correct for the 'enclosed' energy.

The energy at any point, due to a sphere at $x=-d$ moving with a velocity V along z; is, using polar coordinates as before, $x=\rho\cos\theta$, etc.,—

$$\frac{V^2a^6}{8}\left\{3\cos^2\phi\frac{(1-\cos^2\theta)\rho^2}{(2\rho d\cos\theta+\rho^2+d^2)^4}+\frac{1}{(\ ,,\ )^3}\right\}.$$

The integral of this in $\rho\sin\theta d\phi$, round a quadrant, is,—

$$\frac{V^2a^6\pi\rho\sin\theta}{128d^2}\left\{\frac{-3}{(2\rho d\cos\theta+\rho^2+d^2)^2}+\frac{6\rho^2+14d^2}{(\ ,,\ )^3}+\frac{-3\rho^4+6\rho^2d^2-3d^4}{(\ ,,\ )^4}\right\}.$$

The integral of this in $\rho d\theta$, is,—

$$\frac{V^2a^6\pi\rho}{256d^3}\left\{\frac{-3}{(2\rho d\cos\theta+\rho^2+d^2)}+\frac{3\rho^2+7d^2}{(\ ,,\ )^2}+\frac{-\rho^4+2\rho^2d^2-d^4}{(\ ,,\ )^3}\right\}.$$

The value of this from 0 to π, is,—

$$\frac{V^2a^6\pi}{8}\rho^2\frac{2\rho^2+d^2}{(d^2-\rho^2)^4}.$$

The integral of which to ρ, is,—

$$\frac{V^2a^6\pi}{128}\left\{\frac{\rho^5+8\rho^3d^2-\rho d^4}{d^2(d^2-\rho^2)^3}+\frac{1}{2d^3}\log\frac{d+\rho}{d-\rho}\right\}.$$

The value of this from 0 to a, multiplied by 4, is the energy enclosed in a sphere radius a, and distant d from the moving sphere, or,—

$$\frac{V^2a^3\pi}{3}\left\{\frac{e^6}{2}+\frac{9}{5}e^8, \text{ etc.}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(83)$$

is our 'enclosed' energy.

Subtracting this from (81), we have, finally, for spheres in parallel direct,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left\{1+\frac{3}{2}e^3-\frac{9}{8}e^5+\frac{3}{4}e^6\right\}, \quad\ldots\ldots\ldots\ldots(84)$$

$$\frac{dE}{dd}=-\frac{V^2a^6\pi}{d^4}\left\{3-\frac{15}{4}e^2+3e^3\right\}, \quad\ldots\ldots\ldots\ldots(85)$$

and subtracting it from (82), we have for spheres in parallel reversed,—

$$\frac{E}{2}=\frac{V^2a^3\pi}{3}\left\{1-\frac{3}{2}e^3+\frac{9}{8}e^5+\frac{3}{4}e^6\right\}, \quad\ldots\ldots\ldots\ldots(86)$$

$$\frac{dE}{dd}=\frac{V^2a^6\pi}{d^4}\left\{3-\frac{15}{4}e^2-3e^3\right\}. \quad\ldots\ldots\ldots\ldots(87)$$

(85) represents an attraction and (87) a repulsion for reasons similar to those given in the two-dimensional problem: figs. 11 and 12 can be considered as applying, so far as descriptive purposes are concerned, to this three-dimensional case as well. The symmetrical distortion required to make these diagrams correct representations of the three-dimensional lines of flow, would signify a change in the amount of the force; but, of course, no alteration in its sign.

The negative term in (84) does not imply a negative constraint: the whole constraint is, of course, positive; being in fact equivalent to a plane boundary $x=0$: it is our arbitrary method of measuring to the centre of the sphere which requires a correction that may be positive or negative according to circumstances. The distance between the centres of two spheres is not the effective or real distance between the spherical surfaces.

It is particularly to be noticed that,—unlike the result for the two-dimensional problem,—in three dimensions moving spheres exert about twice the force when in sequence that they exert at the same distance when in parallel.

Concerning our neglected 'sink-sources'; it is not difficult to shew, by following a method similar to that pursued in obtaining the 'mean velocity,' that the discharge parallel to x passing through an element of the permeable spherical surface is the common factor $\dfrac{Va^2 d\theta d\phi}{4} e^4$ multiplied into,—

$$\{(6-15e^2)\cos\theta+(-24e+90e^3)\cos^2\theta+(-6+90e^2)\cos^3\theta \\ +(24e-300e^3)\cos^4\theta-75e^2\cos^5\theta+210e^3\cos^6\theta\}\cos\phi:$$

that parallel to y being the same factor multiplied into,—

$$\{6e-15e^3-30e^2\cos\theta+(-12e+135e^3)\cos^2\theta+60e^2\cos^3\theta \\ +(6e-225e^3)\cos^4\theta-30e^2\cos^5\theta+105e^3\cos^6\theta\}(\cos\phi-\cos^3\phi):$$

and the neglected discharge parallel to z; obtained by subtracting from the total z discharge passing through the element, that discharge due to the 'mean velocity' of which we have already taken account; being the same factor multiplied by,—

$$\left\{\begin{aligned}&(4e-10e^3-20e^2\cos\theta+(-8e+90e^3)\cos^2\theta+40e^2\cos^3\theta\\&\quad+(4e-150e^3)\cos^4\theta-20e^2\cos^5\theta+70e^3\cos^6\theta)\frac{3}{2}\cos^3\phi\\&\quad+\left(\frac{3}{4}e-\frac{145}{32}e^3+(6-15e^2)\cos\theta+\left(-\frac{63}{4}e+\frac{1825}{32}e^3\right)\cos^2\theta\right.\\&\quad+(-6+50e^2)\cos^3\theta+\left(15e-\frac{525}{4}e^3\right)\cos^4\theta-35e^2\cos^5\theta\\&\quad\left.+\frac{315}{4}e^3\cos^6\theta\right)\cos\phi.\end{aligned}\right\}$$

No one of these contains a lower power of e than e^4; and therefore its 'solitary' energy can contain no term lower than e^8: and, as we have before shewn, cannot compound with the lines of flow due to any moving sphere, doublet or 'sink-source' to produce additional energy of as low an order as e^6. Their neglect cannot, therefore, vitiate our result so far as shewn; but their rotational effect is too interesting to pass without notice.

The plane xz being one of symmetry, no rotation about an axis in this plane is possible, any rotation must be about an axis perpendicular to it; that is to say, in this three-dimensional case, about y. The discharge parallel to y can therefore produce no rotation.

The moment of the x discharge of the element of area in question about y, is obtained by multiplying that discharge by $a\sin\theta\cos\phi$; giving, on integration to ϕ from 0 to $\frac{\pi}{2}$ and to θ from 0 to $\frac{\pi}{2}$ and π to $\frac{\pi}{2}$, because the area is reversed in the lower octant,—

$$\frac{Va^3e^4\pi}{16}\{3+5e^2\}. \qquad\text{.....................(87}a\text{)}$$

The moment of the neglected z discharge of the same element about y, is obtained by multiplying that discharge by $-a\cos\theta$; the negative sign being prefixed because positive z discharge in the first octant means negative moment about y. Upon integrating this product to ϕ from 0 to $\frac{\pi}{2}$ and to θ from

0 to π; because the area is not reversed in this case; we obtain,—

$$\frac{Va^3e^4\pi}{16}\left\{-3+\frac{15}{4}e^2\right\}. \qquad \text{(87b)}$$

These results applying to one quadrant, and all quadrants round x conspiring, we have a net resultant moment about y,—

$$\frac{35\,Va^3\pi e^6}{16}. \qquad \text{(87c)}$$

High as is the order of e involved and insignificant as the effect may be at great distances, the existence of this rotation is amply sufficient to explain atomic polarity. Tempting, however, as such considerations are, they lie beside our present object.

It may be useful, for reference, to collect here a few of the more characteristic results obtained for the two, and for the three-dimensional problem. In the case of forces we here suppose the distance between the moving particles great compared with their diameters. The formulæ are all either in two dimensions or in cylindrical coordinates, except when y appears: then we have the ordinary three-dimensional rectangular coordinates.

	II.	III.
Solitary Particle—		
Velocity Potential	$-\dfrac{Va^2\cos\theta}{r}$	$-\dfrac{Va^3\cos\theta}{2r^2}$
Line of flow	circle, $c=\dfrac{r}{\sin\theta}$	oval, $c=\dfrac{r}{\sin^2\theta}$
„ ratio length to width	1	$\dfrac{\sqrt{27}}{4}$
Two Particles in Sequence—		
Trace of orthogonal locus	$(x\pm b)^2+z^2=2b^2$	$x^2\pm x\sqrt{9b^2-z^2}+2(z^2-b^2)=0$
„ length $\perp$ b of $-^{ve}$ enclosure	$2b$	$3b$
Determinant	circle, $x^2+z^2=b^2$	oblate spheroid, $\dfrac{x^2}{2b^2}+\dfrac{y^2}{2b^2}+\dfrac{z^2}{b^2}=1$
	line, $x=0$	line, $\left.\begin{matrix}x=0\\y=0\end{matrix}\right\}$
	line or circle, $x^2+z^2=\infty$	sphere or plane, $x^2+y^2+z^2=\infty$
Velocity due to one, at site of other	$+Ve^2$	$+Ve^3$
Force	$\dfrac{4V^2a^4\pi}{d^3}$	$\dfrac{6V^2a^6\pi}{d^4}$
Two Particles in Parallel—		
Trace of orthogonal locus on xz	$(z\pm b)^2+x^2=2b^2$	$2z^2\pm z\sqrt{9b^2-x^2}+x^2-b^2=0$
„ length $\perp$ b of $-^{ve}$ enclosure	$2b$	$\dfrac{3b}{2}$
Determinant	circle, $x^2+z^2=b^2$	ellipse, $\left.\begin{matrix}\dfrac{x^2}{b^2}+\dfrac{2z^2}{b^2}=1\\y=0\end{matrix}\right\}$
	line, $z=0$	plane, $z=0$
	circle or line, $x^2+z^2=\infty$	circle or line, $\left.\begin{matrix}x^2+z^2=\infty\\y=0\end{matrix}\right\}$
Velocity due to one, at site of other	$-Ve^2$	$\dfrac{-Ve^3}{2}$
Force	$\dfrac{4V^2a^4\pi}{d^3}$	$\dfrac{3V^2a^6\pi}{d^4}$
$\dfrac{\text{Force, or 'velocity at site,' or length orthogonal; in sequence}}{\text{Force, or 'velocity at site,' or length orthogonal; in parallel}}$	± 1	± 2

VII

THE DIMENSIONS OF SPACE

Is it too much to hope that any reader who has not already discarded these pages, may be free from that unreasoning confidence in his perceptive faculties which,—excusable in the uneducated,—is too often to be found among those whose knowledge and experience should long ago have taught them that, fallible as reason sometimes proves to be, few things are more deceptive than our 'senses'?

No attempt will be here made to adduce those numerous minor evidences concerning the existence of the higher dimensions of space which occur even in the elementary branches of mathematics; many such can be supplied by anyone capable of appreciating their value: though it may be not unnecessary to point out that the occurrence of so-called 'imaginary' quantities in algebra,—which are incomprehensible except when viewed as vectors,—by no means *in itself* implies the existence of the fourth dimension. Our inability to extract the square root of minus one is primarily due to our unwarranted action in assigning the same symbol to the product of two negative, that we do to the product of two positive quantities.

The greatest of all the evidences in favour of the existence of the higher dimensions of space is one which,—while appearing most puny to a casual glance,—grows ever in proportion to the time during which it is subjected to examination: there is no reason why the dimensions of space should be limited to three.

The simplest event we observe; every problem in geometry; every expression of a truth in pure mathematics; every fact,

whether concrete or abstract, of which we fancy ourselves assured; provided only, it contains numerical coefficients, exponents, dimensions or finite terms; proves upon examination to be but a particular case of a more general expression of truth, obtainable by substituting generalized for particular coefficients *and* making a suitable alteration in the function or expression concerned. We may be quite unable to determine *what* alterations must be made in the function in order that we may obtain a generalized expression which,—when certain numerical values are substituted for the generalized coefficients,—may reduce to the expression of that particular item of truth upon which we first stumbled; but there is no doubt possible as to the existence of such generalized comprising truths. Indeed there are evidently an infinite number of such generalized expressions, of all of which the original is a special case: just as a circle is a special case of an ellipse, as well as a special case,—to select from among our own muttons,—of $\sin^n \theta = \frac{r}{c}$.

Perhaps no single elementary discovery of mathematicians can rival in universal practical application that fact concerning the relation which exists between the lengths of the sides of a triangle, fortunate enough to own an angle of ninety degrees, with which Euclid delighted our budding intelligence. Who has not,—while still entranced in the contemplation of that beautiful flower presented to him by the patient gardener of his earliest play-ground,—acknowledged that an acutely, if short felt want, was satisfied upon that dear preceptor's soon proceeding to exhibit his power of demonstrating an inclusive proposition concerning the sides of less orthodox triangles? Later the student may, perhaps, himself have found expressions applicable to all polygons, comprising both the former as special cases of successive grade: and learnt that such a process is obviously without end.

Concerning our inability to perceive with our physical senses,—or even picture clearly in the mind,—a fourth dimension of space; we should, obviously, first inquire how we perceive, or think we perceive, the three dimensions with

which it is customary to credit that portion of the universe which immediately surrounds us.

When a child is born he is in the position of one, quite ignorant of telegraphic instruments,—to say nothing of codes,—who suddenly opens his eyes in a closed room on the walls of which are furiously ticking five 'needle' instruments. That within a few months, without extraneous assistance, he is able to attach intelligible meanings to the oscillations of those needles, is one of the most astonishing episodes in his existence. The feat is evidently rendered just not impossible by the fact that several, if not all of the needles, usually respond simultaneously to one and the same impulse. It is by the coordination of our senses that we are able to interpret,—or fancy we interpret,—the various impressions conveyed to our brain.

Conceive an unhappy being born into this world with eyesight as his only sense, and affected with a paralysis which included even the sphincter of his crystalline lens. Suppose that in some marvelous manner he still succeeded in attaching some concrete meaning to the various impressions made at different points upon his retina. He would perceive relative motion among these objects, but his ideas on this subject would necessarily be confined to up or down and right or left. He might notice that certain objects at times would wither, become blurred and ultimately vanish; but the idea that they were approaching or receding, would be as far from his conception as the existence of a direction simultaneously perpendicular to the vertical and horizontal would,—even if suggested,—be beyond his comprehension. We perceive and fancy we comprehend the third dimension of space, simply because we can stretch out the arm and feel.

If then we can shew that there is any reason for supposing that if a fourth dimension exist, we may very well be situated under such circumstances that motion in that direction is impossible to us considered as machines, the objection that we do not perceive and are unable to comprehend the fourth dimension, absolutely ceases to retain any validity as an argument *against* its existence.

Consider a few of the functions which we have been investigating: firstly in two dimensions: Velocity potential $-\frac{Va^2\cos\theta}{r}$: line of flow circle $c=\frac{r}{\sin\theta}$: circles in sequence direct attract as the inverse cube of the distance; in parallel direct, attract as the same power of the distance multiplied by the same constant coefficient as in the former case. Secondly in three dimensions: Velocity potential $-\frac{Va^3\cos\theta}{2r^2}$: line of flow oval $c=\frac{r}{\sin^2\theta}$: spheres in sequence direct attract as the inverse fourth power of the distance; in parallel direct, attract as the same power of the distance but with half the numerical value of the force which they exert when in sequence.

Now if we assume that the four-dimensional line of flow is $c=\frac{r}{\sin^3\theta}$, we find that the orthogonal for 'spheres' in sequence has three times the dimension perpendicular to b which characterises the corresponding locus for 'spheres' in parallel. This would shew that the force in four dimensions between 'spheres' moving in sequence is three times the force between them when moving in parallel: and that the velocity potential for four dimensions is $\phi=-\frac{Va^4\cos\theta}{3r^3}$. But we need not here assume anything.

It is perfectly clear why the velocity potential for two dimensions has the first power of r in its denominator. Whatever the motive may be that causes motion at any point in the plane layer of fluid, it obviously has its origin at the moving circle; and just as obviously its intensity must vary inversely as any length, area or volume over which it is evenly spread. Consider two straight lines at a small inclination to each other, drawn as radii from the origin. Clearly all parallel lines,—whatever their inclination to these radii may be,—have intercepts between these lines proportional in length to the distance of those intercepts from the origin: therefore the intensity of the motive in any direction varies inversely as the distance of the locality from the origin. Similarly in the case of three-dimensional motion; in adding another

dimension we make the 'area' as broad as the length was, and is, long; therefore the space over which the motive is diffused varies as r^2, and therefore the intensity of the motive as r^{-2}. Similarly, when we add another dimension, each of the three dimensions of the volume over which the motive has now to act increases as the distance r, therefore the intensity of the motive varies as r^{-3}. Thus it is obvious that the velocity potential for a moving particle in four dimensions must vary as r^{-3}.

Again: in two dimensions, whether we move along z or along x, the *rate* at which the intensity of the motive varies is the same. We have only one dimension to right and left, the detruding lines of flow, being enclosed between parallel walls, find no more rapidly extending freedom in one direction than in another. It therefore follows that there is no difference between the numerical coefficients of x velocities and of z velocities.

In three dimensions, however, the lines of flow of the fluid pushed forward and aside by the moving sphere, *radiate* in every direction from the line of motion. When we move, therefore, away from the line of motion, freedom is gained crossways as rapidly as it is gained sideways; while it is still gained sideways as rapidly as when we moved along z. Therefore the coordinate representing distance from the axis of z has greater voice in the reduction of velocity than has the coordinate parallel to z: so the velocity potential, being of the form $F(Va)\dfrac{z}{K(x^2+z^2)^{\frac{3}{2}}}$, we see that the coefficient K must be, if not 2, at least some quantity greater than that which was its value for two dimensions.

Similarly in four dimensions, having added another dimension to the distance between radiating planes of flow, in place of gaining only breadth proportional to x by leaving the axis of z, we gain in breadth *and* in length as compared with the rate at which we gain freedom by moving along z. Therefore the value of the coordinate z in reducing the velocity is still less powerful, as compared with other dimensions, than it was in the three-dimensional problem: therefore the coefficient K in

the denominator for four dimensions, whether 3 or not, *must* be something greater that 2.

Similarly for space of n dimensions we see that in the velocity potential $\frac{F(Va)\cos\theta}{K_{n-1}\,r^{n-1}}$, the value of K_{n-1} must be greater than the value of K_{n-2}, which stands for the coefficient in the case of $n-1$ dimensions. And therefore the ratio between the force of attraction in the case of particles moving in sequence direct, and the attraction between particles moving in parallel direct, constantly *increases* with the increase in n.

Now consider the effect of these facts upon such motion of material particles as we find universally prevalent throughout as much of the universe as we have power to investigate. A particle of matter on the earth's surface near the equator, for example, may,—and probably will,—have a certain velocity with reference to the earth's surface. This may be that of a wave, or any casual motion or wind drift; its direction may be reversed in a few seconds, it will seldom be maintained unaltered for hours: its maximum amount may be taken as that of a hurricane, or say one hundred miles an hour. The particle is at the same time moving with the earth's surface due east with a velocity, reversed every twelve hours, of one thousand miles an hour. When these velocities are compounded, the effect,—especially over any considerable interval of time,—will scarcely be distinguishable in direction from that of the last mentioned. Again, with the whole earth, the particle possesses a velocity, reversed every six months, of no less than sixty-five thousand miles an hour: still more completely swamping the effect of both velocities before mentioned. Similarly there can be no reasonable doubt that the velocity of the solar system,* relatively to the milky way, enormously

* That which is given in some works on Astronomy as the velocity of the Sun, being derived from observations on the 'proper motions' of adjacent stars, is not the velocity here in question. All adjacent stars of course participate in the orbital motion of the Sun about the centre of the Galaxy. Relative 'proper motion' in such case measures little more than mutual perturbation.

exceeds that of the earth round the sun: even the moon has thirty times the velocity relative to the sun that it has with reference to the earth. Lastly, the velocity of the milky way with reference to the rest of our three-dimensional bit of the infinite universe, must be enormously greater than this last.

It is evident, therefore, that there is throughout each such sub-universe as the milky way, in any universe of a given dimension, one great velocity possessed by all its component material particles; which, for all intents and purposes, may be taken as unalterable throughout countless ages, both in direction and amount; and of such magnitude that the effects of all subordinate motions when compared with the effect of this one grand movement, are utterly inappreciable. This velocity we shall call the 'great' velocity of matter for the dimension in question.

Now, as we have seen, whatever the dimension above the second, spheres in sequence attract each other with a greater force than do spheres in parallel. It therefore follows that in process of time all clusters of particles in n dimensions will tend to flatten themselves out into $n-1$ dimensions: the 'plane' of which will be perpendicular to the n dimensional 'great' velocity; just as the milky way is approaching the form of a disc at right angles to its great velocity. Moreover, as we have seen, the ratio between these two different attractions increases with n; it therefore follows that this result will be achieved in n dimensions before it has been completed in $n-1$ dimensions. So, in an infinitely infinite universe which has existed from all infinity, at any given epoch there must exist sub-universes of every possible number of dimensions.

Now suppose for a moment that all particles of our universe were thus 'flattened' out by the acceleration of following and retardation of leading particles in sequence, until all our atoms were in a two-dimensional surface. However complex the motions and combinations between the atoms in this two-dimensional surface might be, none of their resultants could possess a component perpendicular to their

plane: it would be impossible to construct a machine capable of moving its arm into the third dimension, simply because all force acting between the particles lay in their plane.

It is true that whatever energy there *might* be available over and above the kinetic energy of the motions of particles *in* that plane, would be drawn from the three-dimensional great velocity which all alike possessed in a direction perpendicular to that plane: and supposing we *could* draw upon that energy it would reduce that 'great' three-dimensional velocity. Still the action which effected this, supposing such action to exist, would act on all neighbouring particles in proportion to their proximity, and therefore would at most but locally and temporarily deform the surface without projecting particles out of what was locally the plane of the universe: and such deformation would gradually spread as a wave over all the rest of the surface, tending in time to readjust all again into one plane; which would then, however, be moving in the third dimension with an infinitesimally smaller velocity than it possessed before we drew upon that source of energy.

As a matter of fact we find ourselves in a three-dimensional 'corner' of the infinite universe; and, as above pointed out, are therefore as machines incapable of motion in the fourth or any higher dimension *relatively* to our immediate surroundings: although we, and the whole of our three-dimensional universe, *are* moving in the fourth dimension; and it is from the energy of this four-dimensional great velocity that all energy now existing in our universe, was ultimately drawn. Thus, being incapable of relative movement in the fourth dimension, we cannot perceive it.

Since all motion and configuration in our universe is thus the result of the present *and* past motion in higher dimensions: what is the resultant action *now* existent between one ultimate particle and another ultimate particle in our three-dimensional universe?

First and obviously there is the attraction varying inversely as the fifth power of the distance, which is due to the motion 'in parallel' under that great velocity proper to the fourth

dimension, which is now common to all these particles: but this is not all. During countless ages, while the acceleration on a rearward sphere in sequence and the retardation on a leading sphere have been gradually bringing both into one four-dimensional 'plane' with their neighbours; the attraction between them *in* that 'plane' due to their motion in parallel, has also been acting upon them without intermission. This force,—perhaps one third of the attraction in sequence,—is still of exactly the same *order* of magnitude; and must have produced results which are at least comparable with the present existing inverse fifth.

If we sought the resultant force and detailed configuration consequent upon such action, the 'problem of three bodies' would indeed be child's play in comparison with the question before us; but the principle involved is, as usual, of perfect simplicity. Whatever the complication of motion, configuration and force may be; the resultant is necessarily the sum of the individual component effects.

Given, then, what may be called an infinite number of like particles, which have through all time been attracting each other with a force varying inversely as the fifth power of the distance: required the consequent mutual action between a single pair situated now at a given distance apart; such action being the result, so far as these two are concerned, of *all* the interactions between *all* the particles under the specified action.

Let there be n particles; and suppose each divided into $n-1$ equal portions, each of unit mass: all except one particular particle A, which we are for the moment considering.

Conceive the whole attraction which existed between A and any other particle B, to have been concentrated upon one single unit portion of that particle B; to the exclusion of all its other unit portions: and imagine, in the same manner, the remaining $n-2$ unit portions of B to have been similarly exclusive in their attentions; each to some one particular particle of the universe.

Let $\frac{K}{r^5}$ be the force which has been acting throughout all

time between A and its appropriate unit portion of B; which we may call b.

The excess of potential energy in b (due to its distance from A) when at infinity, over that it possesses now, when at distance d, is $+K\int_d^\infty \frac{dx}{r^5}=\frac{K}{4d^4}$. If v is the resultant velocity of b, its kinetic energy is $\frac{v^2}{2}$, which must be equal to the loss of potential energy during its development, or $v=\frac{\sqrt{2K}}{2d^2}$ towards A.

Similarly, the resultant effect of every one of the other $n-2$ particles on A is that of a unit mass moving towards A at a velocity $v=\frac{\sqrt{2K}}{2d^2}$: and, since we must suppose n infinite, and these component portions to have started from an infinite distance in all directions and at *all* differences of infinite distance; it follows that the aggregate effect upon A at any moment of present time, is *equivalent to* that of an infinite number of units of mass spread uniformly over space around A, and extending so far as the group of particles constituting our three-dimensional universe extends; each unit moving towards A with a velocity $v=\frac{\mu}{d^2}$: that is to say A is, in effect, a three-dimensional sink.

The attraction varying with the inverse fifth power, coexists; but being of the same order of magnitude as to its numerical or constant coefficient, it is inappreciable compared with the force which varies as the inverse squares, except when the particles are within molecular distances from each other.

Although it is obviously objectionable to introduce any speculative matter among conclusions which profess to be necessary consequences of those entities we assumed as our initial data; the mention of one hypothesis here appears unavoidable if we are not to leave our train of reasoning incomplete. The validity of our conclusions does not, however, depend upon the correctness of the particular hypothesis in question; but simply upon the possibility of that which the hypothesis pretends to explain. That is to say, the

truth of any other hypothesis would answer our purpose equally well, provided it shewed to be possible what this is designed to demonstrate.

It may have been noticed that in speaking of the force exerted by the medium upon a pair of moving particles in a universe of any specified number of dimensions, we tacitly ignored, for the time being, all higher dimensions. Thus, while investigating the attraction between moving spheres in three dimensions we assumed that the medium was bounded by our three-dimensional universe: while it may reasonably be objected, that if space possess n dimensions our particles are moving in space of n dimensions, and should therefore experience only that pressure which is due to motion in n-dimensional space and not that due to motion in three-dimensional space.

That our assumption was justifiable, appears capable of proof without the introduction of any hypothesis, but to particularize suppositions will avoid excessive circumlocution.

Let us then suppose, for a moment, that absolute space possesses no more than n dimensions, and that this n-dimensional space is pervaded by a continuous medium, throughout which are scattered myriads of particles, primal or first atoms; and that all these atoms possess relatively to the medium one common great velocity.

It is directly demonstrable that two of these atoms in sequence will attract each other with a force $n-1$ times that exerted on two similar atoms at the same distance apart in parallel. In process of time, therefore, all of the atoms if finite in number, or all of any finite group if we consider the total infinite, will be 'flattened out' into $n-1$ dimensions.

During the great interval of time necessary to effect this, certain of these first atoms will have united to form some sort of molecules which we may call second atoms; while the rest will, being condensed into $n-1$ dimensions, constitute what from an n-dimensional point of view is a thin layer of something resembling our idea of an ordinary fluid; a medium ultimately granular in structure. The 'second atoms,' moving about in this layer, *would* be moving in a medium confined

to $n-1$ dimensions; and their interactions would truly be those due to the motion of particles in an $n-1$ dimensional fluid.

The medium of each succeeding universe would be of grosser texture than that of the last; but the 'atom' of each new universe would also be proportionately larger than that of the old.

APPENDIX

ATOMIC FORCES

It may be not inappropriate here to roughly indicate how one result we have obtained may be applied to the explanation of some natural forces other than gravitation. What follows is for the most part pure hypothesis; and in its present crude form is obviously open to many objections: though the majority of those objections which first occur to the mind lose much of their force upon detailed examination.

In chap. VI. we found that if two particles were moving in parallel, with a velocity V in the direction of $+z$; the line joining their centres being taken as the axis of x; each exerted on the other a rotational moment about the axis of y, equal to,

$$\pm \frac{35 Va^3\pi}{16} e^6.$$

If the velocity V along z be the four-dimensional 'great velocity,' this axis of rotation y being perpendicular to z, lies wholly in our universe; but may have any orientation in the three dimensional plane perpendicular to x.

Any two adjacent particles, supposed uninfluenced by others, therefore possess opposite polarity: the positive rotation was obtained for that particle which was at $x=+b$; for that at $x=-b$ a rotation in the opposite direction would have been obtained.

Whether we suppose the particles to possess mass of their own or not, they will be accelerated in their rotation until the integral pressure of the elemental discharges upon the retreating elements of area of their surfaces is zero: after which, angular velocity about y will remain constant.

Any two such particles though stationary in our universe will therefore still possess, either in the medium or in the medium and in themselves, as moment of momentum, what compared with our standards would be an enormous amount of energy.

Actually pairs of particles are not so isolated; but there will always be a certain net resultant effect; a certain direction round which a certain rotation is taking place. Whether this rotation will be inappreciable or very great will depend upon circumstance, and especially upon what we choose to designate as a 'particle' for the time being.

Thus, adopting usually accepted ideas, a gas such as hydrogen, if at uniform temperature and pressure throughout, considered in bulk is perfectly homogeneous; a cubic nch in one part of its volume contains as many molecules as in another; the average distance between molecules is the same, and each molecule is like another. Molecules near the centre of such a mass can therefore possess no such rotation and polarity as here considered: the effect of an adjacent particle on one side neutralising the effect of an equally adjacent particle on the other.

It is quite otherwise if the particle we are considering is a Daltonian atom. The propinquity between two 'atoms' of hydrogen in the molecule is of an altogether higher order than that existing between the molecules of the gas. These will therefore be strongly polarised and possess a great store of energy due to this rotation alone.

If we consider the 'atom' as made up of many hundreds of electrons revolving round a common centre in, for simplicity, two spherical shells; the electrons in the two shells will be oppositely polarised.

To return to the atom: if a pair of these, such as a molecule of hydrogen, has a velocity very high for three dimensions, although doubtless minute compared with the great four-dimensional velocity on which the angular momentum and main polarity of each atom depends, there will be a not negligible moment tending to rotate the atom similarly about some other axis inclined to the main axis of polarity. This

will produce a slow precessional motion of the axis in four dimensions; but also an extremely rapid though minute nutation of this axis in the particle, or rather of the particle's position on the axis. As this particle is not a true uniform spherical shell, but a congeries of electrons, this vibration of its orientation will be promptly communicated to the medium, and we have radiant energy in the form of lines of flow the velocity in which fluctuates backwards and forwards, the lines of flow themselves vibrating transversely;—light, electro-magnetic waves, or whatever we choose to call them. The fundamental period of these waves will be simply that of one revolution of the atom about its main axis. This period of revolution depends upon the effective diameter of the hydrogen atoms and their distance apart in the molecule; and will therefore be characteristic of this element.

The dimensions and distances apart of the atoms in other gases, though differing in numerical values, will doubtless be of the same order of magnitude: other atoms will therefore vibrate in different, but not widely dissimilar periods.

Now consider a solid; and picture it as a large number of little circles at equal distances apart; and picture the boundary of our solid as the linear periphery of a large circle within which all these little circles lie. Every little circle, particle or molecule of the solid, which is not close to the boundary, will possess no appreciable amount of this rotation; the effect of each adjacent particle in this respect being cancelled by the opposite effect of a similar and equally distant particle on the opposite side. Otherwise, however, will it be with the particles around the periphery,—on the surface of our solid. These, each acting upon its neighbour *in* the surface, will annul their mutual action; but the action upon them of the layer of particles next inwards from the surface will not be cancelled; and therefore all the particles on the surface will have, in one sense, a like polarity; rotating about directions in the surface.

Take two such similar bodies and bring them into what is commonly called contact,—close, though scarcely molecular, proximity. A certain amount of their polarity will now

be annulled, and the surface molecules concerned will approximate somewhat to the condition of internal particles. Upon subsequent separation, however, everything will revert to its former condition; we shall perceive no change in the bodies.

If we take two different solids, the molecules in one will presumably be at a different distance apart, of different mass and of a different size from that which in each case characterises the particles of the other body. When the surface molecules of these are brought together they will not have mutually equal action upon each other; the different systems of lines of flow of the sink-sources causing these rotations, with the different angular momenta and different moments of inertia of the two sets of particles, acting from particle to particle will strike some kind of average throughout the double body during contact. If now we separate the two bodies they will not relapse into their former conditions; one will have more momentum or somewhat differently directed polarity or differ in both these or some other respects from its original condition; and the other will have experienced an exactly opposite or complementary change. We shall perceive a difference in the condition of each body, and the two differences will be of the same magnitude but of opposite sign: this is the phenomenon of 'contact electricity.'

Instead of thus slightly approximating minute portions of large masses, suppose we dash violently a molecule of chlorine against a molecule of hydrogen: they will rebound; or more probably describe quasi hyperbolic orbits about each other, the epoch of 'perihelia' in which we call the instant of contact. While they were in contact, that same action which we saw occurring when two dissimilar large masses touched, would take effect; but more completely. Each chlorine molecule would lose something which the hydrogen molecule would gain, or vice versâ. Upon separation, if the two hydrogen atoms remain together and the two chlorine atoms remain together, the two molecules will each be in a new state; the change in one being opposite to the change in the other; there will be some sort of difference of potential between them: the system consisting of the two molecules

will have gained potential energy; though in the molecules themselves potential energy may have decreased.

If, on the other hand, when the four atoms separate each half of the double molecule should consist of one hydrogen atom and one chlorine atom, there will be no difference between the two new molecules; there will be no difference of potential between them as molecules. If, then, molecular gain in the former path exceed possibly existent atomic loss, since at the moment of contact both these paths lie open before the atoms, by the principle of stability they will choose the latter: this is chemical combination. Any fluid, for instance, under the action of gravity in various connected vessels, will choose that path of all paths open to it, motion along which causes its potential with respect to the earth most rapidly to diminish: and, as in this rough analogy the lost potential energy appears as heat; so in the case of the molecules of hydrogen chloride, the potential energy which their components would have possessed with reference to each other if they could have been directed along the other path, appears as heat and light: these, or some other form of radiant energy, being the usual result of chemical combination.

So far we have been considering our particles as spheres; at least as congeries of electrons or the like which, taken as a whole, are symmetrical about a point. Possible arrangements are, however, obviously not limited to such symmetry. Our atom may, from a four-dimensional point of view, be disc-like in form; resembling the solar system, or perhaps more probably Saturn's rings.

The rotational effect of particles in parallel would be quite different here. It may be remembered that this effect to which we are attributing so profound an influence upon atomic forces, was the net difference between two opposite effects; one due to what we called the z component, the other due to what we called the x component. That the latter preponderated over the former depended upon the fact that their effect was taken over a spherical surface. Since we are concerning ourselves here with only qualitative results,

there is no necessity to integrate our sink-source discharge components over a highly oblate spheroid in order to ascertain the general nature of their effect upon a disc-like particle: the action's sign is obvious.

If we steer a canoe due south towards the centre of a whirlpool which is rotating from east through north to west, the bow will be swept round violently towards the west: the canoe will experience a negative rotation. The z component acting over the length of the canoe altogether overpowers the x component which has but the narrow beam upon which to exert its pressure. If on the other hand we attempt to punt the canoe broadside on towards the centre, it will rotate in a positive direction; the x component having the whole length of the canoe on which to act drives the east end north and the west end south.

If the particle be disc-like, its solitary position of stability will be at right angles to the great velocity, *i.e.* to z. Starting from this position the z component will impart a negative rotation, but before the particle has rotated through $\frac{\pi}{2}$, the x component will utterly overpower the z component and drive it back again. It may be expected to vibrate about its position of stable equilibrium; which will be about 45° from the axis of x, between $+x$ and $+z$. At about right angles to this position, or in the second quadrant, there will be a second position in which the effects of the z and x components balance; but this latter will be a position of unstable equilibrium.

When the particle is in its undeflected position in the plane xy, all its lines of flow cut that plane,—which represents our three-dimensional universe,—at right angles; we perceive no motion in the medium there. When the particle is in its inclined position of stability the lines of flow cut the axis of x obliquely, having on both sides one and the same directioned component along x; from $+x$ to $-x$. This can be easily seen by considering the two extreme electrons, as we may call them, on the periphery of the disc; that which is most depressed below the plane xy and that which is most

elevated above it. This constitutes a stationary magnetic field. Such a particle cannot, however, thus exist in an isolated position: its inclination is due to propinquity of another, whose inclination and magnetic field are opposed to those of the first.

If, however, we consider a massive body, all the surface particles will have a balance of inclination out of the plane xy beyond that which is neutralised by that lesser inclination of the next inner layer; just as we found the spherical particle on the surface of a solid had a net resultant rotation.

Although the normal effect of the sink-sources on a disc-like particle would be such deflection or reciprocating motion; still, as we saw, the integral over a spherical surface shewed a preponderating balance in favour of the x component and positive rotation. It therefore follows that if the disc particle do pass $\frac{\pi}{2}$ with a finite velocity it will commence to revolve continuously in the positive direction; though, of course with quadrantal fluctuations in its angular velocity.

Now we saw that the stationary deflection of disc particles on the surface of a solid, if uncancelled in their effect, produced a magnetic field in which the lines of flow cut the axis of x obliquely. Near the 'magnet' the lines of flow would be nearly parallel to x, at a distance they would be nearly at right angles to this direction. By moving along x a body consisting of disc particles we expose these to a veering current; which, if it conspire with their direction of possible continuous rotation, will set them in continuous rotation: this is magneto-electric induction.

The quadrantal fluctuations will be transmitted through the surrounding medium as corresponding vibrations in the lines of flow: these vibrations and those rotations, when varying from point to point, constitute the electric current.

If the impulse which helped the first few particles over their dead point be but momentary, the rotating particles in starting the next particles whose axes are in a line with their own will, by reaction, lose that extra angular momentum required for continuous rotation, and will consequently be

again reduced to vibration of an amplitude less than π: a single 'wave' of current will pass round the circuit.

Those vibrations depending upon the quadrantal fluctuations in angular velocity, which were transmitted through the medium, would not be absorbed by spherical particles but pass freely through a surrounding medium containing only such spherical particles; but upon encountering the surface of a mass of disc particles they would be quickly absorbed in starting these particles: thus we have the difference between conductors and non-conductors.

Contrary to those ideas which are embodied in the old-fashioned conception of an electric current as a something flowing through a copper wire and incapable of flowing through gutta-percha or air; we see that a 'conductor' is really that which resists the transfer of the fundamental motive,—quadrantal vibrations in the lines of flow,—by absorbing this passing energy in the form of angular momentum in its particles; while the 'non-conductor' or dielectric is that which permits the radiation to traverse space in which its particles are scattered, unimpeded.

This transfer of energy between rotating disc particles on one body and non-rotating particles on another, of course takes place only during acceleration of the former; or at the advancing face of the wave in the medium which their change of motion produced. When the front of the wave has passed, if there be no acceleration, the mean direction of and velocity in the line of flow at any given point in space is constant: the veering current due to the quadrantal fluctuations is of far too short a period to allow its action during one half period to accelerate a particle from its position of rest until it has passed its dead point, before that action has been reversed. Thus a current in one conductor induces a current in a second only during acceleration, or change of distance: this is voltaic induction.

Similarly, in the case of a finite charged conductor, all statical induction takes place during charging or during relative motion.

On the other hand, whether rotating disc particles are

being accelerated or not, provided the effect be not cancelled, they imply a magnetic field. It is true that while the disc is in the first and third quadrants the lines of flow where they cut the plane xy have, as we have seen, a negative x component; while, during the disc's passage through the second and fourth quadrants this component will be positive: but these effects by no means cancel. The rotating disc spends a much longer period in the quadrants which contain its position of unstable equilibrium than in the others. In the same way a pendulum in continuous rotation in a vertical plane takes much longer to describe the upper than to describe the lower half of its circular path.

Thus electro-magnetic induction is not necessarily dependent upon acceleration of current or relative motion; though the word 'induction' here is a misnomer: a magnetic field is as necessary a concomitant of a rotating disc particle as it is of a deflected stationary disc particle. We can, however, properly speak of electro-magnetic induction upon a magnetic substance in the field of a current; for here an effect is produced upon the magnetic substance; the planes of its disc particles being 'blown' more or less towards perpendicularity to the lines of flow in the current's field. The nodes of the inclined orbits being thus shifted, discordant molecular fields are brought into accord: a molar field has been 'induced.'

In the case of rotating discs which are incapable of changing the orientation of their axes, polarity in the molar field is produced only by variation in the amount of rotation found existing as we consider successive positions along a given direction: by a current as distinguished from a statical charge.

Now let us take a finite and isolated mass of 'conducting' substance; picturing it as a circle in the plane of xy. The volume of the conductor is represented by the area of the circle, and its surface by the circle's linear periphery. We have to content ourselves with such a lame illustration as this because we are unable to picture four dimensions simultaneously; and must therefore club two of our three dimensions into one.

Let us start a line of particles on the circumference rotating, and continue to supply energy until the current has run round the circuit; and we have a closed ring of continuously rotating disc particles: this is a 'statically charged' conductor. The absolute rotations in space of the discs at opposite ends of a diameter are in opposite directions, but everywhere the direction with reference to the inside and outside of the surface is the same; and upon this depends the sign of the 'charge.'

Now approach a second conductor to one portion of the circumference of the first. The quadrantal vibrations passing across the dielectric will set up continuous rotation on the near side of the second conductor in the same absolute direction as that of the particles from which they originated; that is to say induce a charge of the opposite sign on the facing surface. Similarly acting across the body they will induce a charge of the same sign as that of the first conductor, on the far side of the second: this is 'statical induction.'

If we increase the charge on our first conductor we somewhat increase each induced charge on the second; but the opposite signed charges on the second conductor constitute opposed rotations taken in circuit round the sides, so that but the excess of induction on the near side over that on the further remains; we can induce but a small charge on the 'plates of our condenser.'

If now we connect the back of the second conductor to an indefinitely large uncharged conductor, the rotating discs on the back at once give up their angular momentum and cease rotating. If, then, similar transfer of energy from the front of the plate were impossible, we should no longer be prevented by counter induction from increasing indefinitely the momentum stored upon the opposed faces. This actually appears to be brought about by the very removal of the counter induction at once permitting the charge on the face of the first plate, together with the 'free' charge on its back, to bring the rotations of the particles facing each other across the dielectric into perfect accord with each other; so that each is a perfect 'negative constraint' to the other. That is

to say, one and the same system of circulatory lines of flow, embracing both rotating particles, completely supplies the rotary motional requirements of each. These lines of flow probably are enclosed within the surface of what may be described as a crossed vortex ring. Outside this ring there are no lines of flow due to the rotary motion; and therefore no means of transferring that rotation to other particles: this represents the 'bound charge.'

The free charge on the first conductor maintains this rather precarious condition of equilibrium, by supplying the energy required to restore equality disturbed by that unavoidable loss which is due to imperfect realisation of this ideal state; being itself recouped by half the leakage.

If we remove this free charge, without allowing a corresponding free charge to exist on the second conductor, the equilibrium rendered unstable by leakage quickly runs down: the condenser 'discharges itself.'

If we persist in supplying energy in the form of angular momentum to the rotating disc particles on the first plate, the face particles equally share with those of the second plate, until the increasing distance of the molecule's constituents from their common centre so far lessens the central attraction that they begin to leave the system in quasi-hyperbolic orbits.

Those particles which are separated across the dielectric by the shortest interval will have their constituents revolving in the longest orbits, and will consequently be the first to commence disintegration. This stream, of atoms or electrons as the case may be, breaking through the crossed rings unites the two conductors by a direct bridge; along which the two rotations meet in violently opposed torque. The rotations cancel each other, their energy appears as heat, light and similar radiations, sound, a little chemical action and a little, so called, mechanical force: this is the 'disruptive' discharge.

As an observed fact, the surface particles of an ordinary massive body have naturally no definite orientation in their arrangement in the surface, unless the mass in question is

the whole or part of one large crystal: even crystalline masses have the axes of their component crystals arranged in no common direction.

Excepting single crystals, therefore, a mass of disc particles would not by the inclinations of its particles be able to produce a magnetic field of molar magnitude without rearrangement of its particles; since the effects of no appreciable majority would conspire: a conductor is not necessarily a magnetic body.

A magnetic body must be composed of disc particles capable of at least a certain amount of rotation about the axis of y, unless the necessary inclination pre-existed in the particles; and so these are probably capable of making a complete revolution: a magnetic substance, if not a crystal, is probably a conductor: but this is not all.

The structure of the molecules and their connections must be such that the orbital planes can be rotated until their lines of nodes are parallel to each other and similarly directed; possessing also a certain stability in this position.

Since so many requirements must be fulfilled by the particles of a magnetic substance, one would expect markedly magnetic substances to be rare, and their particles highly complex. Decidedly our best criterion concerning the complexity of an invisible structure is the number of distinct periods in which it is capable of vibrating: perhaps the most complex of all spectra is that of iron.

The rotational moment acts, in the case of a massive body, as we have seen, upon only the surface molecules; a few layers from the surface the molecule, though possibly capable of rotation, will not continue to rotate, even if in closed chains, when left to itself: 'statical electricity' is noticeable on only the surface of charged bodies.

That motive radiation which constitutes the electric current in the dielectric, readily sets up rotation in the surface layer of the 'wire.' If continued it will also set rotating the inner layers of 'neutral' molecules; but this will require time, the process being an effort as against inertia: the penetration of electric current into the mass of a conductor is a compara-

tively gradual process. If the motive be rapidly alternated in direction, a hollow tube should be as efficient a 'wire' as a solid cylinder.

Through a non-conductor, on the other hand, the quadrantal vibrations, although not nutational in their nature, are freely transmitted; and their period being the first octave of the fundamental nutational vibration of the disc molecules concerned, they are presumably conveyed at a similar velocity: 'electrical displacement' is transmitted through the dielectric with the velocity of light.

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